5章　三角関数

§3　加法定理とその応用　(p.153〜p.160)

問1

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \boldsymbol{\frac{\sqrt{6}+\sqrt{2}}{4}}\end{aligned}$$

$$\begin{aligned}\tan 75^\circ &= \tan(45^\circ + 30^\circ)\\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\cdot\frac{1}{\sqrt{3}}}\\ &= \frac{\left(1 + \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}{\left(1 - \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}\\ &= \frac{\sqrt{3}+1}{\sqrt{3}-1}\\ &= \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}\\ &= \frac{3+2\sqrt{3}+1}{3-1}\\ &= \frac{4+2\sqrt{3}}{2} = \boldsymbol{2+\sqrt{3}}\end{aligned}$$

〔別解〕

$$\begin{aligned}\tan 75^\circ &= \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\frac{\sqrt{6}+\sqrt{2}}{4}}{\frac{\sqrt{6}-\sqrt{2}}{4}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}} = \frac{(\sqrt{6}+\sqrt{2})^2}{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})}\\ &= \frac{6+2\cdot 2\sqrt{3}+2}{6-2}\\ &= \frac{8+4\sqrt{3}}{4} = \boldsymbol{2+\sqrt{3}}\end{aligned}$$

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \boldsymbol{\frac{\sqrt{6}-\sqrt{2}}{4}}\end{aligned}$$

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}\\ &= \boldsymbol{\frac{\sqrt{6}+\sqrt{2}}{4}}\end{aligned}$$

$$\begin{aligned}\tan 15^\circ &= \tan(45^\circ - 30^\circ)\\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}\\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\cdot\frac{1}{\sqrt{3}}}\\ &= \frac{\left(1 - \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}{\left(1 + \frac{1}{\sqrt{3}}\right)\times\sqrt{3}}\\ &= \frac{\sqrt{3}-1}{\sqrt{3}+1}\\ &= \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}\\ &= \frac{3-2\sqrt{3}+1}{3-1}\\ &= \frac{4-2\sqrt{3}}{2} = \boldsymbol{2-\sqrt{3}}\end{aligned}$$

〔別解〕

$$\begin{aligned}\tan 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{\sqrt{6}-\sqrt{2}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{(\sqrt{6}-\sqrt{2})^2}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}\\ &= \frac{6-2\cdot 2\sqrt{3}+2}{6-2}\\ &= \frac{8-4\sqrt{3}}{4} = \boldsymbol{2-\sqrt{3}}\end{aligned}$$

問2

$$\begin{aligned}\text{与式} &= \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}}\\ &= \frac{\tan\theta + 1}{1 - \tan\theta\cdot 1}\\ &= \boldsymbol{\frac{1+\tan\theta}{1-\tan\theta}}\end{aligned}$$

問 3

α は第 2 象限の角だから，$\cos\alpha < 0$

よって

$$\cos\alpha = -\sqrt{1-\sin^2\alpha}$$
$$= -\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)^2}$$
$$= -\sqrt{1-\frac{1}{3}}$$
$$= -\sqrt{\frac{2}{3}} = -\frac{\sqrt{6}}{3}$$

β は第 4 象限の角だから，$\cos\beta > 0$

よって

$$\cos\beta = \sqrt{1-\sin^2\beta}$$
$$= \sqrt{1-\left(-\frac{\sqrt{5}}{6}\right)^2}$$
$$= \sqrt{1-\frac{5}{36}}$$
$$= \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{6}$$

したがって

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$= \frac{\sqrt{3}}{3}\cdot\frac{\sqrt{31}}{6} - \left(-\frac{\sqrt{6}}{3}\right)\cdot\left(-\frac{\sqrt{5}}{6}\right)$$
$$= \frac{\sqrt{93}}{18} - \frac{\sqrt{30}}{18}$$
$$= \boldsymbol{\frac{\sqrt{93}-\sqrt{30}}{18}}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= -\frac{\sqrt{6}}{3}\cdot\frac{\sqrt{31}}{6} + \frac{\sqrt{3}}{3}\cdot\left(-\frac{\sqrt{5}}{6}\right)$$
$$= -\frac{\sqrt{186}}{18} - \frac{\sqrt{15}}{18}$$
$$= \boldsymbol{-\frac{\sqrt{186}+\sqrt{15}}{18}}$$

問 4

$0 < \alpha < \frac{\pi}{2}$，$0 < \beta < \frac{\pi}{2}$ の辺々を加えると

$$0 < \alpha+\beta < \frac{\pi}{2} + \frac{\pi}{2}$$

すなわち，$0 < \alpha+\beta < \pi \cdots$ ①

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$
$$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}$$
$$= \frac{\frac{5}{6}}{1-\frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = \mathbf{1}$$

また，①より，$\alpha+\beta = \boldsymbol{\frac{\pi}{4}}$

問 5

α は第 2 象限の角だから，$\sin\alpha > 0$

よって

$$\sin\alpha = \sqrt{1-\cos^2\alpha}$$
$$= \sqrt{1-\left(-\frac{4}{5}\right)^2}$$
$$= \sqrt{1-\frac{16}{25}}$$
$$= \sqrt{\frac{9}{25}} = \frac{3}{5}$$

したがって

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right)$$
$$= \boldsymbol{-\frac{24}{25}}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$= 2\cdot\left(-\frac{4}{5}\right)^2 - 1$$
$$= \frac{32}{25} - 1 = \boldsymbol{\frac{7}{25}}$$

$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha}$$
$$= \frac{-\frac{24}{25}}{\frac{7}{25}} = \boldsymbol{-\frac{24}{7}}$$

問 6

$$\cos^2\frac{\pi}{8} = \cos^2\frac{\frac{\pi}{4}}{2}$$
$$= \frac{1+\cos\frac{\pi}{4}}{2}$$
$$= \frac{1+\frac{1}{\sqrt{2}}}{2}$$
$$= \frac{\left(1+\frac{\sqrt{2}}{2}\right)\times 2}{2\times 2}$$
$$= \frac{2+\sqrt{2}}{4}$$

$\cos\dfrac{\pi}{8} > 0$ であるから

$$\cos\frac{\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}} = \boldsymbol{\frac{\sqrt{2+\sqrt{2}}}{2}}$$

問 7

$\dfrac{\pi}{2} < \alpha < \pi$ より，$\dfrac{\pi}{4} < \dfrac{\alpha}{2} < \dfrac{\pi}{2} \cdots$①

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} = \frac{1-\left(-\frac{1}{9}\right)}{2} = \frac{\frac{10}{9}}{2} = \frac{5}{9}$$

①より，$\sin\dfrac{\alpha}{2} > 0$ であるから

$$\sin\frac{\alpha}{2} = \sqrt{\frac{5}{9}} = \boldsymbol{\frac{\sqrt{5}}{3}}$$

また

$$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} = \frac{1+\left(-\frac{1}{9}\right)}{2} = \frac{\frac{8}{9}}{2} = \frac{4}{9}$$

①より，$\cos\dfrac{\alpha}{2} > 0$ であるから

$$\cos\frac{\alpha}{2} = \sqrt{\frac{4}{9}} = \boldsymbol{\frac{2}{3}}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} = \boldsymbol{\frac{\sqrt{5}}{2}}$$

問 8

（1）与式 $= \dfrac{1}{2}\{\cos(3\theta+5\theta)+\cos(3\theta-5\theta)\}$

$$= \frac{1}{2}\{\cos 8\theta + \cos(-2\theta)\}$$

$$= \boldsymbol{\frac{1}{2}(\cos 8\theta + \cos 2\theta)}$$

（2）与式 $= -\dfrac{1}{2}\{\cos(7\theta+2\theta)-\cos(7\theta-2\theta)\}$

$$= -\frac{1}{2}(\cos 9\theta - \cos 5\theta)$$

$$= \boldsymbol{\frac{1}{2}(\cos 5\theta - \cos 9\theta)}$$

問 9

（1）与式 $= 2\sin\dfrac{4\theta+2\theta}{2}\cos\dfrac{4\theta-2\theta}{2}$

$$= 2\sin\frac{6\theta}{2}\cos\frac{2\theta}{2}$$

$$= \boldsymbol{2\sin 3\theta\cos\theta}$$

（2）与式 $= 2\cos\dfrac{3\theta+5\theta}{2}\cos\dfrac{3\theta-5\theta}{2}$

$$= 2\cos\frac{8\theta}{2}\cos\frac{-2\theta}{2}$$

$$= 2\cos 4\theta\cos(-\theta)$$

$$= \boldsymbol{2\cos 4\theta\cos\theta}$$

問 10

（1）$y = \sqrt{1^2+1^2}\sin(x+\alpha)$

$$= \sqrt{2}\sin(x+\alpha)$$

ここで，$\cos\alpha = \dfrac{1}{\sqrt{2}}$，$\sin\alpha = \dfrac{1}{\sqrt{2}}$ より，$\alpha = \dfrac{\pi}{4}$

よって，$\boldsymbol{y = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}$

（2）$y = \sqrt{1^2+(\sqrt{3})^2}\sin(x+\alpha)$

$$= 2\sin(x+\alpha)$$

ここで，$\cos\alpha = \dfrac{1}{2}$，$\sin\alpha = -\dfrac{\sqrt{3}}{2}$ より，$\alpha = -\dfrac{\pi}{3}$

よって，$\boldsymbol{y = 2\sin\left(x-\frac{\pi}{3}\right)}$

問 11

$$y = \sqrt{2^2+3^2}\sin(x+\alpha)$$

$$= \sqrt{13}\sin(x+\alpha)$$

ただし，$\cos\alpha = \dfrac{2}{\sqrt{13}}$，$\sin\alpha = \dfrac{3}{\sqrt{13}}$ である．

ここで，$-1 \leqq \sin(x+\alpha) \leqq 1$ であるから

$$-\sqrt{13} \leqq \sqrt{13}\sin(x+\alpha) \leqq \sqrt{13}$$

よって，最大値は $\boldsymbol{\sqrt{13}}$，最小値は $\boldsymbol{-\sqrt{13}}$

∎