BASIC

48　曲線 C 上において，$x=2t,\ y=t^2,\ z=\dfrac{1}{3}t^3$ であるから

$$\frac{dx}{dt}=2,\quad \frac{dy}{dt}=2t,\quad \frac{dz}{dt}=t^2$$

（1）

$$\begin{aligned}\frac{ds}{dt}&=\left|\frac{d\boldsymbol{r}}{dt}\right|\\&=\sqrt{2^2+(2t)^2+(t^2)^2}\\&=\sqrt{t^4+4t^2+4}\\&=\sqrt{(t^2+2)^2}\\&=|t^2+2|=t^2+2\end{aligned}$$

よって

$$\begin{aligned}与式&=\int_C(xy+z)\,ds\\&=\int_0^1\left(2t\cdot t^2+\frac{1}{3}t^3\right)\frac{ds}{dt}\,dt\\&=\int_0^1\frac{7}{3}t^3(t^2+2)\,dt\\&=\frac{7}{3}\int_0^1(t^5+2t^3)\,dt\\&=\frac{7}{3}\left[\frac{1}{6}t^6+\frac{1}{2}t^4\right]_0^1\\&=\frac{7}{3}\left(\frac{1}{6}+\frac{1}{2}-0\right)\\&=\frac{7}{3}\cdot\frac{2}{3}=\boldsymbol{\frac{14}{9}}\end{aligned}$$

（2）

$$\begin{aligned}与式&=\int_C(xy+z)\,dy\\&=\int_0^1\left(2t\cdot t^2+\frac{1}{3}t^3\right)\frac{dy}{dt}\,dt\\&=\int_0^1\frac{7}{3}t^3\cdot 2t\,dt\\&=\frac{14}{3}\int_0^1 t^4\,dt\\&=\frac{14}{3}\left[\frac{1}{5}t^5\right]_0^1\\&=\frac{14}{3}\left(\frac{1}{5}-0\right)\\&=\frac{14}{3}\cdot\frac{1}{5}=\boldsymbol{\frac{14}{15}}\end{aligned}$$

49　曲線 C 上で，$\boldsymbol{a}=(\cos t,\ \sin t,\ t)$

また，$\dfrac{d\boldsymbol{r}}{dt}=(-\sin t,\ \cos t,\ 1)$

よって，求める戦績分の値は

$$\begin{aligned}\int_C\boldsymbol{a}\cdot d\boldsymbol{r}&=\int_C\boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt\\&=\int_0^{2\pi}\{\cos t\cdot(-\sin t)+\sin t\cdot\cos t+t\cdot 1\}\,dt\\&=\int_0^{2\pi}t\,dt\\&=\left[\frac{1}{2}t^2\right]_0^{2\pi}\\&=\frac{1}{2}\cdot(2\pi)^2=\boldsymbol{2\pi^2}\end{aligned}$$

50（1）　曲線 C_1 上で

$$\begin{aligned}\boldsymbol{a}&=(t+(1-t)\cdot 0,\quad (1-t)^3+t\cdot 0,\quad t(1-t))\\&=(t,\ (1-t)^3,\ t(1-t))\end{aligned}$$

また，$\dfrac{d\boldsymbol{r}}{dt}=(1,\ -1,\ 0)$

よって

$$\begin{aligned}与式&=\int_{C_1}\boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt\\&=\int_0^1\{t\cdot 1+(1-t)^3\cdot(-1)+t(1-t)\cdot 0\}\,dt\\&=\int_0^1\{t-(1-3t+3t^2-t^3)\}\,dt\\&=\int_0^1(t^3-3t^2+4t-1)\,dt\\&=\left[\frac{1}{4}t^4-t^3+2t^2-t\right]_0^1\\&=\frac{1}{4}-1+2-1=\boldsymbol{\frac{1}{4}}\end{aligned}$$

（2）　与式 $=\displaystyle\int_{C_1}\boldsymbol{a}\cdot d\boldsymbol{r}+\int_{C_2}\boldsymbol{a}\cdot d\boldsymbol{r}$ であるから，$\displaystyle\int_{C_2}\boldsymbol{a}\cdot d\boldsymbol{r}$ を求める．　曲線 C_2 上で

$$\begin{aligned}\boldsymbol{a}&=(\cos t+\sin t\cdot 0,\quad \sin^3 t+\cos t\cdot 0,\quad \cos t\cdot\sin t)\\&=(\cos t,\ \sin^3 t,\ \cos t\sin t)\end{aligned}$$

また，$\dfrac{d\boldsymbol{r}}{dt}=(-\sin t,\ \cos t,\ 0)$

よって

$$\begin{aligned}\int_{C_2}\boldsymbol{a}\cdot d\boldsymbol{r}&=\int_{C_2}\boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt\\&=\int_0^{\frac{\pi}{2}}\{\cos t\cdot(-\sin t)+\sin^3 t\cdot\cos t\\&\qquad\qquad+\cos t\sin t\cdot 0\}\,dt\\&=\int_0^{\frac{\pi}{2}}(-\cos t\sin t+\sin^3 t\cos t)\,dt\\&=\int_0^{\frac{\pi}{2}}\sin t\cos t(1-\sin^2 t)\,dt\\&=\int_0^{\frac{\pi}{2}}\sin t\cos t\cdot\cos^2 t\,dt\\&=\int_0^{\frac{\pi}{2}}\sin t\cos^3 t\,dt\\&=\left[\frac{1}{4}\cos^4 t\right]_0^{\frac{\pi}{2}}\\&=\frac{1}{4}(0^4-1^4)=-\frac{1}{4}\end{aligned}$$

したがって

$$\begin{aligned}与式&=\int_{C_1}\boldsymbol{a}\cdot d\boldsymbol{r}+\int_{C_2}\boldsymbol{a}\cdot d\boldsymbol{r}\\&=\frac{1}{4}+\left(-\frac{1}{4}\right)=\boldsymbol{0}\end{aligned}$$

51　$C_1,\ C_2,\ C_3$ で囲まれた範囲を D とすると

$D:\ 0\leqq x\leqq 1,\quad 0\leqq y\leqq 1-x$

また，$\dfrac{\partial}{\partial x}(x+y)=1,\quad \dfrac{\partial}{\partial y}(xy^2)=2xy$ であるから，グリーンの定理より

$$
\begin{aligned}
\text{与式} &= \iint_D (1-2xy)\,dx\,dy \\
&= \int_0^1 \left\{ \int_0^{1-x} (1-2xy)\,dy \right\} dx \\
&= \int_0^1 \Big[y - xy^2 \Big]_0^{1-x} dx \\
&= \int_0^1 \{(1-x) - x(1-x)^2\}\,dx \\
&= \int_0^1 \{1 - x - x(1-2x+x^2)\}\,dx \\
&= \int_0^1 (-x^3+2x^2-2x+1)\,dx \\
&= \Big[-\frac{1}{4}x^4 + \frac{2}{3}x^3 - x^2 + x \Big]_0^1 \\
&= -\frac{1}{4} + \frac{2}{3} - 1 + 1 \\
&= \frac{-3+8}{12} = \boldsymbol{\frac{5}{12}}
\end{aligned}
$$

C_1 上で，$x=t,\ y=0$ であるから，$\dfrac{dx}{dt}=1,\ \dfrac{dy}{dt}=0$

よって

$$
\begin{aligned}
&\int_{C_1} \{xy^2\,dx + (x+y)\,dy\} \\
&= \int_{C_1} xy^2\,dx + \int_{C_1} (x+y)\,dy \\
&= \int_0^1 t\cdot 0^2 \frac{dx}{dt}dt + \int_0^1 (t+0)\frac{dy}{dt}dt \\
&= \int_0^1 0\cdot 1\,dt + \int_0^1 t\cdot 0\,dt \\
&= 0+0=0
\end{aligned}
$$

C_2 上で，$x=1-t,\ y=t$ であるから，$\dfrac{dx}{dt}=-1,\ \dfrac{dy}{dt}=1$

よって

$$
\begin{aligned}
&\int_{C_2} \{xy^2\,dx + (x+y)\,dy\} \\
&= \int_{C_2} xy^2\,dx + \int_{C_2} (x+y)\,dy \\
&= \int_0^1 (1-t)\cdot t^2 \frac{dx}{dt}dt + \int_0^1 \{(1-t)+t\}\frac{dy}{dt}dt \\
&= \int_0^1 (t^2-t^3)\cdot(-1)\,dt + \int_0^1 1\cdot 1\,dt \\
&= \int_0^1 (t^3-t^2+1)\,dt \\
&= \Big[\frac{1}{4}t^4 - \frac{1}{3}t^2 + t \Big]_0^1 \\
&= \frac{1}{4} - \frac{1}{3} + 1 \\
&= \frac{3-4+12}{12} = \frac{11}{12}
\end{aligned}
$$

C_3 上で，$x=0,\ y=1-t$ であるから，$\dfrac{dx}{dt}=0,\ \dfrac{dy}{dt}=-1$

よって

$$
\begin{aligned}
&\int_{C_3} \{xy^2\,dx + (x+y)\,dy\} \\
&= \int_{C_3} xy^2\,dx + \int_{C_3} (x+y)\,dy \\
&= \int_0^1 0\cdot(1-t)^2 \frac{dx}{dt}dt + \int_0^1 \{0+(1-t)\}\frac{dy}{dt}dt \\
&= \int_0^1 0\cdot 0\,dt + \int_0^1 (1-t)\cdot(-1)\,dt \\
&= \int_0^1 (t-1)\,dt \\
&= \Big[\frac{1}{2}t^2 - t \Big]_0^1 \\
&= \frac{1}{2} - 1 = -\frac{1}{2}
\end{aligned}
$$

以上より

$$
\begin{aligned}
\text{与式} &= \int_{C_1+C_2+C_3} \{xy^2\,dx + (x+y)\,dy\} \\
&= 0 + \frac{11}{12} + \left(-\frac{1}{2}\right) \\
&= \frac{11-6}{12} = \boldsymbol{\frac{5}{12}}
\end{aligned}
$$

52　$\dfrac{\partial \boldsymbol{r}}{\partial u} = (1,\ 0,\ -1)$，$\dfrac{\partial \boldsymbol{r}}{\partial v} = (0,\ 1,\ -1)$ であるから

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} \\
&= 0\,\boldsymbol{i} + 0\,\boldsymbol{j} + \boldsymbol{k} - \{-\boldsymbol{i} - \boldsymbol{j} + 0\,\boldsymbol{k}\} \\
&= \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} \\
&= (1,\ 1,\ 1)
\end{aligned}
$$

これより，$\left| \dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v} \right| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$D: 0 \leqq u \leqq 1,\quad 0 \leqq v \leqq 1$ とすると

$$
\begin{aligned}
\int_S \varphi\,dS &= \int_S xyz\,dS \\
&= \iint_D uv(1-u-v)\cdot\sqrt{3}\,du\,dv \\
&= \sqrt{3}\int_0^1 \left\{ \int_0^1 (uv - u^2v - uv^2)\,dv \right\} du \\
&= \sqrt{3}\int_0^1 \Big[\frac{1}{2}(u-u^2)v^2 - \frac{1}{3}uv^3 \Big]_0^1 du \\
&= \sqrt{3}\int_0^1 \left\{ \frac{1}{2}(u-u^2) - \frac{1}{3}u \right\} du \\
&= \sqrt{3}\int_0^1 \left(\frac{1}{6}u - \frac{1}{2}u^2 \right) du \\
&= \sqrt{3}\Big[\frac{1}{12}u^2 - \frac{1}{6}u^3 \Big]_0^1 \\
&= \sqrt{3}\left(\frac{1}{12} - \frac{1}{6} \right) \\
&= \sqrt{3}\cdot\left(-\frac{1}{12} \right) = \boldsymbol{-\frac{\sqrt{3}}{12}}
\end{aligned}
$$

53　$\dfrac{\partial \boldsymbol{r}}{\partial u} = (1,\ 1,\ 2u)$，$\dfrac{\partial \boldsymbol{r}}{\partial v} = (-1,\ 1,\ 2v)$ であるから

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 2u \\ -1 & 1 & 2v \end{vmatrix} \\
&= 2v\,\boldsymbol{i} - 2u\,\boldsymbol{j} + \boldsymbol{k} - \{2u\,\boldsymbol{i} + 2v\,\boldsymbol{j} - \boldsymbol{k}\} \\
&= (-2u+2v)\boldsymbol{i} + (-2u-2v)\,\boldsymbol{j} + 2\,\boldsymbol{k} \\
&= (-2u+2v,\ -2u-2v,\ 2)
\end{aligned}
$$

$\boldsymbol{n}$ の z 成分が正であるから，$\boldsymbol{n} = \dfrac{\dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v}}{\left| \dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v} \right|}$

これより，$\boldsymbol{n}\left| \dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v} \right| = \dfrac{\partial \boldsymbol{r}}{\partial u} \times \dfrac{\partial \boldsymbol{r}}{\partial v}$

また，曲面 S 上で，$\boldsymbol{a} = (u+v,\ u-v,\ u^2+v^2)$ であるから，求める面積分の値は

$$\begin{aligned}
\int_S \boldsymbol{a}\cdot\boldsymbol{n}\,dS &= \iint_D \boldsymbol{a}\cdot\boldsymbol{n}\left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| du\,dv \\
&= \iint_D \boldsymbol{a}\cdot\left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) du\,dv \\
&= \iint_D \{(u+v)(-2u+2v) \\
&\qquad + (u-v)(-2u-2v) \\
&\qquad\qquad + (u^2+v^2)\cdot 2\}\,du\,dv \\
&= 2\iint_D \{(v^2-u^2)-(u^2-v^2)+(u^2+v^2)\}\,du\,dv \\
&= 2\iint_D (3v^2-u^2)\,du\,dv \\
&= 2\int_0^1 \left\{ \int_0^1 (3v^2-u^2)\,dv \right\} du \\
&= 2\int_0^1 \Big[v^3-u^2v \Big]_0^1 du \\
&= 2\int_0^1 (1-u^2)\,du \\
&= 2\left[u-\frac{1}{3}u^3 \right]_0^1 \\
&= 2\left(1-\frac{1}{3}\right) \\
&= 2\cdot\frac{2}{3} = \boldsymbol{\frac{4}{3}}
\end{aligned}$$

54　$\nabla\cdot\boldsymbol{a} = \dfrac{\partial}{\partial x}(x^2y)+\dfrac{\partial}{\partial y}(y^2z)+\dfrac{\partial}{\partial x}(z^2x) = 2xy+2yz+2zx$

よって，求める面積分の値は

$$\begin{aligned}
\int_S \boldsymbol{a}\cdot\boldsymbol{n}\,dS &= \int_V \nabla\cdot\boldsymbol{a}\,dV \\
&= \int_V (2xy+2yz+2zx)\,dx\,dy\,dz \\
&= \int_0^1 \left\{ \int_0^1 \left\{ \int_0^1 (2xy+2yz+2zx)dz \right\} dy \right\} dx \\
&= \int_0^1 \left\{ \int_0^1 \Big[2xyz+xz^2+yz^2 \Big]_0^1 dy \right\} dx \\
&= \int_0^1 \left\{ \int_0^1 (2xy+x+y)\,dy \right\} dx \\
&= \int_0^1 \left[xy^2+xy+\frac{1}{2}y^2 \right]_0^1 dx \\
&= \int_0^1 \left(2x+\frac{1}{2} \right) dx \\
&= \left[x^2+\frac{1}{2}x \right]_0^1 \\
&= 1+\frac{1}{2} = \boldsymbol{\frac{3}{2}}
\end{aligned}$$

55（1）　C 上で

$$\boldsymbol{a} = (-\sin t,\ \cos t,\ 0)$$
$$\frac{d\boldsymbol{r}}{dt} = (-\sin t,\ \cos t,\ 0)$$

よって

$$\begin{aligned}
\int_S (\nabla\times\boldsymbol{a})\cdot\boldsymbol{n}\,dS &= \int_C \boldsymbol{a}\cdot d\boldsymbol{r} \\
&= \int_0^{2\pi} \boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt \\
&= \int_0^{2\pi} \{(-\sin t)^2+\cos^2 t+0\}\,dt \\
&= \int_0^{2\pi} 1\,dt \\
&= \Big[t \Big]_0^{2\pi} = \boldsymbol{2\pi}
\end{aligned}$$

（2）　C 上で

$$\begin{aligned}
\boldsymbol{a} &= (0-2\cdot\sin t,\ \cos t-2\cdot 0,\ \sin t-2\cdot\cos t) \\
&= (-2\sin t,\ \cos t,\ \sin t-2\cos t)
\end{aligned}$$

$$\frac{d\boldsymbol{r}}{dt} = (-\sin t,\ \cos t,\ 0)$$

よって

$$\begin{aligned}
\int_S (\nabla\times\boldsymbol{a})\cdot\boldsymbol{n}\,dS &= \int_C \boldsymbol{a}\cdot d\boldsymbol{r} \\
&= \int_0^{2\pi} \boldsymbol{a}\cdot\frac{d\boldsymbol{r}}{dt}\,dt \\
&= \int_0^{2\pi} \{-2\sin t\cdot(-\sin t)+\cos^2 t+0\}\,dt \\
&= \int_0^{2\pi} (2\sin^2 t+\cos^2 t)\,dt \\
&= \int_0^{2\pi} \{2\sin^2 t+(1-\sin^2 t)\}\,dt \\
&= \int_0^{2\pi} (\sin^2 t+1)\,dt \\
&= \int_0^{2\pi} \left\{ \frac{1-\cos 2t}{2}+1 \right\} dt \\
&= \int_0^{2\pi} \left(\frac{3}{2}-\frac{1}{2}\cos 2t \right) dt \\
&= \left[\frac{3}{2}t-\frac{1}{4}\sin 2t \right]_0^{2\pi} \\
&= \frac{3}{2}\cdot 2\pi = \boldsymbol{3\pi}
\end{aligned}$$

∎