1章　ベクトル解析

§2　スカラー場とベクトル場(p.18～p.28)

問1

（1）$\nabla\varphi=\left(\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial z}\right)=\left(\frac{y}{x},\ \log x,\ \cos z\right)$

$$(\nabla\varphi)_{\mathrm{P}}=\left(\frac{-2}{1},\ \log 1,\ \cos\pi\right)=(\mathbf{-2},\ \mathbf{0},\ \mathbf{-1})$$

（2）$|(\nabla\varphi)_{\mathrm{P}}|=\sqrt{(-2)^2+0^2+(-1)^2}$

$$=\sqrt{4+1}$$

$$=\sqrt{5}$$

よって，

$$\boldsymbol{n}=\frac{1}{\sqrt{5}}(-2,\ 0,\ -1)=\left(-\frac{\mathbf{2}}{\sqrt{\mathbf{5}}},\ \mathbf{0},\ -\frac{\mathbf{1}}{\sqrt{\mathbf{5}}}\right)$$

（3）$(\nabla\varphi)_{\mathrm{P}}\cdot\boldsymbol{k}=\left(-\frac{2}{\sqrt{5}}\right)\cdot(-2)+0\cdot0+\left(-\frac{1}{\sqrt{5}}\right)\cdot(-1)$

$$=\frac{4}{\sqrt{5}}+\frac{1}{\sqrt{5}}$$

$$=\frac{5}{\sqrt{5}}$$

$$=\sqrt{\mathbf{5}}$$

（4）$|\boldsymbol{a}|=\sqrt{2^2+1^2+2^2}=\sqrt{4+1+4}=\sqrt{9}=3$

$\boldsymbol{a}$と同じ向きの単位ベクトルを$\boldsymbol{e}$とすると，

$$\boldsymbol{e}=\frac{1}{3}(2,\ 1,\ 2)$$

よって，求める方向微分係数は

$$(\nabla\varphi)_{\mathrm{P}}\cdot\boldsymbol{e}=\frac{1}{3}\{(-2)\cdot2+0\cdot1+(-1)\cdot2\}$$

$$=\frac{1}{3}(-4+0-2)$$

$$=\frac{1}{3}\cdot(-6)$$

$$=\mathbf{-2}$$

問2

（Ⅰ）

$$左辺=\left(\frac{\partial(\varphi+\psi)}{\partial x},\ \frac{\partial(\varphi+\psi)}{\partial y},\ \frac{\partial(\varphi+\psi)}{\partial z}\right)$$

$$=\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial x},\ \frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial y},\ \frac{\partial\varphi}{\partial z}+\frac{\partial\psi}{\partial z}\right)$$

$$=\left(\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial z}\right)+\left(\frac{\partial\psi}{\partial x},\ \frac{\partial\psi}{\partial y},\ \frac{\partial\psi}{\partial z}\right)$$

$$=\nabla\varphi+\nabla\psi=右辺$$

（Ⅲ）

$$左辺=\left(\frac{\partial f(\varphi)}{\partial x},\ \frac{\partial f(\varphi)}{\partial y},\ \frac{\partial f(\varphi)}{\partial z}\right)$$

$$=\left(f'(\varphi)\cdot\frac{\partial\varphi}{\partial x},\ f'(\varphi)\cdot\frac{\partial\varphi}{\partial y},\ f'(\varphi)\cdot\frac{\partial\varphi}{\partial z}\right)$$

$$=f'(\varphi)\left(\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial z}\right)$$

$$=f'(\varphi)\nabla\varphi=右辺$$

問3

（1）$f(\psi)=\frac{1}{\psi}$とする．

$f'(\psi)=-\frac{1}{\psi^2}$となる．

ここで，勾配の公式Ⅲより

$\nabla f(\psi)=f'(\psi)\nabla\psi$となるから，

$$\nabla\left(\frac{1}{\psi}\right)=-\frac{1}{\psi^2}\nabla\psi$$

（2）勾配の公式Ⅱより

$$左辺=(\nabla\varphi)\frac{1}{\psi}+\varphi\left(\nabla\frac{1}{\psi}\right)\quad (1)より$$

$$=(\nabla\varphi)\frac{1}{\psi}+\varphi\left(-\frac{1}{\psi^2}\nabla\psi\right)$$

$$=\frac{\psi\nabla\varphi-\varphi\nabla\psi}{\psi^2}=右辺$$

問4

（1）

$$\nabla\cdot\boldsymbol{a}=\left(\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right)\cdot(xz^2,\ -2xyz,\ yz)$$

$$=\frac{\partial}{\partial x}(xz^2)+\frac{\partial}{\partial y}(-2xyz)+\frac{\partial}{\partial z}(yz)$$

$$=z^2+(-2xz)+y$$

$$=\boldsymbol{z^2-2xz+y}$$

$$\nabla \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz^2 & -2xyz & yz \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(-2xyz)\right\}\boldsymbol{i}$$

$$-\left\{\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial z}(xz^2)\right\}\boldsymbol{j}$$

$$+\left\{\frac{\partial}{\partial x}(-2xyz) - \frac{\partial}{\partial y}(xz^2)\right\}\boldsymbol{k}$$

$$= \{z - (-2xy)\}\boldsymbol{i} - (0 - 2xz)\boldsymbol{j} + (-2yz - 0)\boldsymbol{k}$$

$$= (z + 2xy)\boldsymbol{i} + 2xz\boldsymbol{j} - 2yz\boldsymbol{k}$$

$$= (\boldsymbol{z + 2xy},\quad \boldsymbol{2xz},\quad \boldsymbol{-2yz})$$

（2）

$$\nabla \cdot \boldsymbol{b} = \left(\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial z}\right)\cdot(x^2 + y^2,\quad y^2 + z^2,\quad z^2 + x^2)$$

$$= \frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial}{\partial y}(y^2 + z^2) + \frac{\partial}{\partial z}(z^2 + x^2)$$

$$= \boldsymbol{2x + 2y + 2z}$$

$$\nabla \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & y^2 + z^2 & z^2 + x^2 \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(z^2 + x^2) - \frac{\partial}{\partial z}(y^2 + z^2)\right\}\boldsymbol{i}$$

$$-\left\{\frac{\partial}{\partial x}(z^2 + x^2) - \frac{\partial}{\partial z}(x^2 + y^2)\right\}\boldsymbol{j}$$

$$+\left\{\frac{\partial}{\partial x}(y^2 + z^2) - \frac{\partial}{\partial y}(x^2 + y^2)\right\}\boldsymbol{k}$$

$$= (0 - 2z)\boldsymbol{i} - (2x - 0)\boldsymbol{j} + (0 - 2y)\boldsymbol{k}$$

$$= -2z\boldsymbol{i} - 2x\boldsymbol{j} - 2y\boldsymbol{k}$$

$$= (\boldsymbol{-2z},\quad \boldsymbol{-2x},\quad \boldsymbol{-2y})$$

問5

（Ⅰ・第１式）

$\boldsymbol{a}$と$\boldsymbol{b}$の成分表示をそれぞれ，

$\boldsymbol{a} = (a_x,\quad a_y,\quad a_z)$，　$\boldsymbol{b} = (b_x,\quad b_y,\quad b_z)$とする．

左辺 $= \nabla \cdot (\boldsymbol{a} + \boldsymbol{b})$

$$= \nabla \cdot (a_x + b_x,\quad a_y + b_y,\quad a_z + b_z)$$

$$= \frac{\partial}{\partial x}(a_x + b_x) + \frac{\partial}{\partial y}(a_y + b_y) + \frac{\partial}{\partial z}(a_z + b_z)$$

$$= \frac{\partial a_x}{\partial x} + \frac{\partial b_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial b_y}{\partial y} + \frac{\partial a_z}{\partial z} + \frac{\partial b_z}{\partial z}$$

$$= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} + \frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\cdot(a_x,\quad a_y,\quad a_z)$$

$$+ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\cdot(b_x,\quad b_y,\quad b_z)$$

$= \nabla \cdot \boldsymbol{a} + \nabla \cdot \boldsymbol{b} =$ 右辺

（Ⅰ・第２式）

左辺 $= \nabla \times (\boldsymbol{a} + \boldsymbol{b})$

$$= \nabla \times (a_x + b_x,\quad a_y + b_y,\quad a_z + b_z)$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x + b_x & a_y + b_y & a_z + b_z \end{vmatrix}$$

$$= \left\{\frac{\partial}{\partial y}(a_z + b_z) - \frac{\partial}{\partial z}(a_y + b_y)\right\}\boldsymbol{i}$$

$$-\left\{\frac{\partial}{\partial x}(a_z + b_z) - \frac{\partial}{\partial z}(a_x + b_x)\right\}\boldsymbol{j}$$

$$+\left\{\frac{\partial}{\partial x}(a_y + b_y) - \frac{\partial}{\partial y}(a_x + b_x)\right\}\boldsymbol{k}$$

$$= \left(\frac{\partial a_z}{\partial y} + \frac{\partial b_z}{\partial y} - \frac{\partial a_y}{\partial z} - \frac{\partial b_y}{\partial z}\right)\boldsymbol{i}$$

$$-\left(\frac{\partial a_z}{\partial x} + \frac{\partial b_z}{\partial x} - \frac{\partial a_x}{\partial z} - \frac{\partial b_x}{\partial z}\right)\boldsymbol{j}$$

$$+\left(\frac{\partial a_y}{\partial x} + \frac{\partial b_y}{\partial x} - \frac{\partial a_x}{\partial y} - \frac{\partial b_x}{\partial y}\right)\boldsymbol{k}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z},\quad \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x},\quad \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$+\left(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z},\quad \frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x},\quad \frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}\right)$$

$= \nabla \times \boldsymbol{a} + \nabla \times \boldsymbol{b} =$ 右辺

（Ⅲ・第１式）

$\nabla\varphi = \left(\dfrac{\partial\varphi}{\partial x},\quad \dfrac{\partial\varphi}{\partial y},\quad \dfrac{\partial\varphi}{\partial z}\right)$とする．

左辺 $= \nabla \times \nabla\varphi$

$$= \left(\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial z}\right)\times\left(\frac{\partial\varphi}{\partial x},\quad \frac{\partial\varphi}{\partial y},\quad \frac{\partial\varphi}{\partial z}\right)$$

$$=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$=\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial z} \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{vmatrix}\boldsymbol{k}$$

$$=\left(\frac{\partial}{\partial y}\frac{\partial \varphi}{\partial z}-\frac{\partial}{\partial z}\frac{\partial \varphi}{\partial y}\right)\boldsymbol{i}-\left(\frac{\partial}{\partial x}\frac{\partial \varphi}{\partial z}-\frac{\partial}{\partial z}\frac{\partial \varphi}{\partial x}\right)\boldsymbol{j}$$

$$+\left(\frac{\partial}{\partial x}\frac{\partial \varphi}{\partial y}-\frac{\partial}{\partial y}\frac{\partial \varphi}{\partial x}\right)\boldsymbol{k}$$

$$=\left(\frac{\partial^2 \varphi}{\partial y \partial z}-\frac{\partial^2 \varphi}{\partial y \partial z}\right)\boldsymbol{i}-\left(\frac{\partial^2 \varphi}{\partial z \partial x}-\frac{\partial^2 \varphi}{\partial z \partial x}\right)\boldsymbol{j}$$

$$+\left(\frac{\partial^2 \varphi}{\partial x \partial y}-\frac{\partial^2 \varphi}{\partial x \partial y}\right)\boldsymbol{k}$$

$$=0\boldsymbol{i}-0\boldsymbol{j}+0\boldsymbol{k}$$
$$=(0,\quad 0,\quad 0)$$

$=\boldsymbol{0}=$ 右辺

（Ⅲ・第 2 式）

$\boldsymbol{a}$の成分表示を$\boldsymbol{a}=(a_x,\quad a_y,\quad a_z)$とする．

$$\nabla\times\boldsymbol{a}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$=\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_y & a_z \end{vmatrix}\boldsymbol{i}-\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ a_x & a_z \end{vmatrix}\boldsymbol{j}+\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ a_x & a_y \end{vmatrix}\boldsymbol{k}$$

$$=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)\boldsymbol{i}-\left(\frac{\partial a_z}{\partial x}-\frac{\partial a_x}{\partial z}\right)\boldsymbol{j}+\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)\boldsymbol{k}$$

$$=\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z},\quad \frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x},\quad \frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)$$

よって，

左辺 $=\nabla\cdot(\nabla\times\boldsymbol{a})$

$$=\left(\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial z}\right)$$

$$\cdot\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z},\quad \frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x},\quad \frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial a_z}{\partial y}-\frac{\partial a_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial a_x}{\partial z}-\frac{\partial a_z}{\partial x}\right)$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial a_y}{\partial x}-\frac{\partial a_x}{\partial y}\right)$$

$$=\frac{\partial^2 a_z}{\partial x \partial y}-\frac{\partial^2 a_y}{\partial z \partial x}+\frac{\partial^2 a_x}{\partial y \partial z}-\frac{\partial^2 a_z}{\partial x \partial y}+\frac{\partial^2 a_y}{\partial z \partial x}-\frac{\partial^2 a_x}{\partial y \partial z}$$

$=0=$ 右辺

問 6

回転の公式（Ⅱ）（Ⅲ）より，

左辺 $=\nabla\times(\varphi\nabla\varphi)$

$$=(\nabla\varphi)\times(\nabla\varphi)+\varphi(\nabla\times\nabla\varphi)$$
$$=\boldsymbol{0}+\varphi\cdot\boldsymbol{0}$$

$=\boldsymbol{0}=$ 右辺

問 7

（1）

$$r=\sqrt{x^2+y^2+z^2}=(x^2+y^2+z^2)^{\frac{1}{2}}$$
$$\frac{1}{r}=(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}\frac{1}{r}=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x=-x\left(\frac{1}{r}\right)^3=-\frac{x}{r^3}$$

$$\frac{\partial}{\partial y}\frac{1}{r}=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2y=-y\left(\frac{1}{r}\right)^3=-\frac{y}{r^3}$$

$$\frac{\partial}{\partial z}\frac{1}{r}=-\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2z=-z\left(\frac{1}{r}\right)^3=-\frac{z}{r^3}$$

よって，

$$\nabla\left(\frac{1}{r}\right)=\left(-\frac{x}{r^3},\quad -\frac{y}{r^3},\quad -\frac{z}{r^3}\right)$$

$$=-\frac{1}{r^3}(x,\quad y,\quad z)=-\frac{\boldsymbol{r}}{r^3}$$

（2）与式を変形し，（1）の結果を用いると，

$$\nabla\cdot\left(\frac{\boldsymbol{r}}{r}\right)=\left(\nabla\left(\frac{1}{r}\right)\right)\cdot\boldsymbol{r}+\frac{1}{r}(\nabla\cdot\boldsymbol{r})$$

$$=-\frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{r}+\frac{1}{r}(\nabla\cdot\boldsymbol{r})$$

$$=-\frac{1}{r^3}(\boldsymbol{r}\cdot\boldsymbol{r})+\frac{1}{r}\{\nabla\cdot\boldsymbol{r}\}$$

$$=-\frac{1}{r^3}|\boldsymbol{r}|^2+\frac{1}{r}\left\{\left(\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y},\quad \frac{\partial}{\partial z}\right)\cdot(x,\quad y,\quad z)\right\}$$

$$=-\frac{1}{r^3}|\boldsymbol{r}|^2+\frac{1}{r}\left(\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial x}\right)$$

$$= -\frac{1}{r^3}|\boldsymbol{r}|^2 + \frac{1}{r}(1+1+1)$$

$$= -\frac{r^2}{r^3} + \frac{3}{r}$$

$$= -\frac{1}{r} + \frac{3}{r}$$

$$= \boldsymbol{\frac{2}{r}}$$

（３）与式を変形し，（１）の結果を用いると，

$$\nabla \times \frac{\boldsymbol{r}}{r} = \left\{\nabla\left(\frac{1}{r}\right)\right\} \times \boldsymbol{r} + \frac{1}{r}(\nabla \times \boldsymbol{r})$$

$$= -\frac{\boldsymbol{r}}{r^3} \times \boldsymbol{r} + \frac{1}{r}(\nabla \times \boldsymbol{r})$$

$$= -\frac{1}{r^3}(\boldsymbol{r} \times \boldsymbol{r}) + \frac{1}{r}(\nabla \times \boldsymbol{r})$$

$$= \boldsymbol{0} + \frac{1}{r}\left\{\left(\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}\right) \times (x, \quad y, \quad z)\right\}$$

$$= \frac{1}{r}\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \frac{1}{r}\left\{\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}\boldsymbol{k}\right\}$$

$$= \frac{1}{r}\left\{\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\boldsymbol{k}\right\}$$

$$= \frac{1}{r}\{(0-0)\boldsymbol{i} - (0-0)\boldsymbol{j} + (0-0)\boldsymbol{k}\}$$

$$= \frac{1}{r}(0, \quad 0, \quad 0)$$

$$= \boldsymbol{0}$$

問８

（１）

$$\frac{\partial \varphi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial^2 \varphi}{\partial x^2} = 2yz$$

$$\frac{\partial \varphi}{\partial y} = x^2 z, \quad \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$$\frac{\partial \varphi}{\partial z} = x^2 y + 8xz, \quad \frac{\partial^2 \varphi}{\partial z^2} = 8x$$

よって，

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$= 2yz + 0 + 8x = \boldsymbol{8x + 2yz}$$

（２）

$$\frac{\partial \varphi}{\partial x} = 2xy + z^2, \quad \frac{\partial^2 \varphi}{\partial x^2} = 2y$$

$$\frac{\partial \varphi}{\partial y} = x^2 + 2\mathrm{yz}, \quad \frac{\partial^2 \varphi}{\partial y^2} = 2\mathrm{z}$$

$$\frac{\partial \varphi}{\partial z} = y^2 + 2zx, \quad \frac{\partial^2 \varphi}{\partial z^2} = 2x$$

よって，

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$= \boldsymbol{2x + 2y + 2z}$$

（３）

$$\frac{\partial \varphi}{\partial x} = \frac{2x}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{2(x^2 + y^2 + z^2) - 2x \cdot 2x}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{-2x^2 + 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial \varphi}{\partial y} = \frac{2y}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{2(x^2 + y^2 + z^2) - 2y \cdot 2y}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2x^2 - 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial \varphi}{\partial z} = \frac{2y}{x^2 + y^2 + z^2}$$

$$\frac{\partial^2 \varphi}{\partial z^2} = \frac{2(x^2 + y^2 + z^2) - 2z \cdot 2z}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 2z^2}{(x^2 + y^2 + z^2)^2}$$

よって，

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$= \frac{-2x^2 + 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2} + \frac{2x^2 - 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2} + \frac{2x^2 + 2y^2 - 2z^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2x^2 + 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^2}$$

$$= \boldsymbol{\frac{2}{x^2 + y^2 + z^2}}$$

（4）

$$\varphi = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\frac{\partial\varphi}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2x$$

$$= -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2\varphi}{\partial x^2} = (-x)'(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$+(-x)\left\{(x^2+y^2+z^2)^{-\frac{3}{2}}\right\}'$$

$$= -(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$-x\left\{-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}\cdot 2x\right\}$$

$$= -\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}+\frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$= \frac{-1}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$+\frac{3x^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{-1\cdot(x^2+y^2+z^2)+3x^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}$$

どの変数で偏微分しても，

変わるのは分子の符号と係数だけなので，

$$\frac{\partial^2\varphi}{\partial y^2} = \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2\varphi}{\partial z^2} = \frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}$$

よって，

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$

$$= \frac{2x^2-y^2-z^2-x^2+2y^2-z^2-x^2-y^2+2z^2}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{0}{(x^2+y^2+z^2)^2\sqrt{x^2+y^2+z^2}}$$

$$= \mathbf{0}$$