Complete Rotation Models and Classification of Linear-phase Generalized LOT

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Abstract

This paper discusses the construction of rotation models of linear-phase generalized LOT (GenLOT) which can express all the orthonormal bases. The bases of GenLOT can be generated by extending the dimensions to both sides of the bases in multi-dimensional orthogonal space, and adding operations of rotation and permutation, etc., which satisfy the 3 conditions: symmetry, orthogonality and norm 1 for every stage. In order to describe all the combinations of these operations succinctly, the finite symmetric permutation group and the sign inversion group for the columns of the basic symmetric matrix are defined. Then, a normal subgroup H extensible to an infinite rotation group is extracted from the direct product G of these groups. Next, all elements of G are classified into 4 residue classes using modulus H, and rotation models are generated by reducing the redundant operations between these stages. As variations of these models are expanded 4 times at every stage, the optimal parameters search must be done efficiently. On the other hand, coding gain, used widely as a measure of coding efficiency, is unaffected by the operations of permutation, sign inversion and reflection of LOT bases. Using these properties, we examined equivalent transformation rules between the stages where the optimal value of coding gain is conserved in 4-dimensional and 6-dimensional rotation models. Also, it is shown that rotation models can be classified into several groups equal to the number of stages using the 4 extracted rules mentioned above.

Keyword

Generalized LOT, Linear phase, Rotation model, Complete construction, Classification, Coding gain, Equivalent transformation rule

1 Introduction

Block coding using orthogonal transforms such as DCT(Discrete Cosine Transform) and sub-band coding using filter banks are used for transform coding of images. To improve the block distortion that occurs when the compression ratio of DCT is increased, Malver et al. proposed LOTs that satisfy the conditions of orthogonality, linear phase, and perfect reconstruction[2]. The method consists of decomposing the DCT basis into even- and odd-order bases and extending the basis length by twice the block size. Furthermore, they proposed methods to reduce the real number of multiplications by exploiting the symmetry of the extended basis[3]. This technique is achieved by substituting the transformations of adjacent blocks with delay elements in the blocks and by introducing butterfly operations of addition and subtraction.

On the other hand, Vetterli et al. showed that LOT and filter banks are equivalent transformations using the so-called polyphase matrix. Furthermore, they proposed methods for designing filters with perfect reconstruction and orthogonality using regularity and paraunitarity of the polyphase matrix, which allows for constant delays[4][5].

In general, the block size was set to an even number in the LOT design to take advantage of the symmetry of the transformations. In filter bank design, this number corresponds to the number of channels or divisions.

Soman et al. generalized the condition and proposed methods (GenLOT) to construct filter banks with an arbitrary number of channels with orthogonality and linear phase[6].

Further extending this approach, Queiroz et al. reported methods for constructing LOTs with a basis that is an integer multiple of the block size[7][8].

On the other hand, Izawa focused on the property that rotation operations in LOTs constitute symmetric rotation groups, and proposed efficient methods for designing multi-dimensional linear phase LOTs using the group property[11]. Furthermore, he also introduced geometric models based on the rotation of the orthogonal transform and reported minimal sets of parameters by which the states of all the bases of the LOT can be represented[12]. By extending this rotation model, he designed generalized LOTs of linear phases whose basis length is expanded to an integer multiple of the block size[13].

As described above, a number of various designs of LOT and optimization methods have already been reported. However, the question of how many optimal solutions exist when the coding gain and the evaluation scale are applied has not been clarified. For example, the Lattice model, which is widely used in the design of generalized LOTs, Each stage consists of a butterfly-like addition/subtraction, a delay element, and 2 orthogonal transformation sections.

The optimization process determines the minimum number of rotation parameters and the parameter values that maximize the coding gain values. In the above optimization, when the orthogonal transform is represented only by a rotation operation, That is, when the value of its determinant is 1, it was considered in detail.

On the other hand, combining a rotation with odd permutations reduces the determinant to -1, however, no rigorous study has been made. Basically, when the determinant of the orthogonal transform section is 1 and -1, the optimal solutions for the basis shape and coding gain take different values.

As the number of stages increases, the number of combinations increases by a power law, more efficient optimization methods are required. However, it is not easy to simplify the redundant operations between stages in the conventional lattice model because it includes a delay element z^{-1}.

The purpose of this study is to introduce rotational models with simple geometric structures and to organize and integrate the corresponding equivalent operations between stages.

In general, the transform matrix of the discrete Fourier transform (DFT) has regularity and symmetry. Using this property, the so-called Fast Fourier Transform (FFT) is derived by reducing the real number of multiplications by introducing butterfly operations of addition and subtraction based on distribution laws such as $A \cdot B \pm A \cdot C = A(B \pm C)$.

In the above rotation model, there is symmetry in the rotation operations of each stage. For the most basic model, integrating the multiplicative part of the rotation using butterfly operations of addition and subtraction, etc., leads to generalized LOTs with conventional lattice structures[13]. First, we sought concise descriptions of rotational models that could represent all states of 4-8 dimensional basis of generalized LOTs. In the next step, finite symmetric permutation groups and symmetric sign inversion groups were derived from the extended symmetry formula.

All operations satisfying the above symmetry equation are represented by the direct product G of their combinations. From this, we extract a normal subgroup H which can be extended to a continuous rotation group, and showed that it can be classified into 4 residue classes (cosets) using H as modulus.

The rotation model is constructed by repeating the operations corresponding to the 4 classes at each stage. As the number of stages increases, the number of combinations increases by a power of 4.

In general, operations such as LOT basis substitution, sign reversal of $\pm$, and mirroring do not change the value of the coding gain, which is widely used as a measure of coding efficiency. Using this property, we derived an equivalent transformation rule between stages whose optimal values are preserved in 4- to 8-dimensional rotation models in order to integrate the redundant operations left between stages[11].

Furthermore, by organizing and integrating the above rotation model using the 4 rules extracted, it is clarified that the final classification is into groups equal to the number of stages[14].

2 Construction of symmetric 4-dimensional orthonormal basis

First, let us review the construction of a symmetric 4-dimensional orthonormal basis[11].

2.1 Basic symmetric matrix of orthonormal basis E_4

All 4-dimensional orthonormal bases that are linear phases can be represented using the (4×4) matrix T_4 shown below.

$$T_4 = \begin{pmatrix} a_1 & a_2 & a_2 & a_1 \\ b_1 & b_2 & b_2 & b_1 \\ c_1 & c_2 & -c_2 & -c_1 \\ d_1 & d_2 & -d_2 & -d_1 \end{pmatrix} \tag{1}$$

Each row of the matrix T_4 corresponds to a basis, and with respect to the vertical symmetry axis between columns 2 and 3.

Rows 1 and 2 are even-symmetric components, rows 3 and 4 are odd-symmetric components.

An normalized orthogonal transform is formed when the following conditions are satisfied,

$$a_1^2 + a_2^2 = \frac{1}{2} \tag{2}$$

$$b_1^2 + b_2^2 = \frac{1}{2} \tag{3}$$

$$c_1^2 + c_2^2 = \frac{1}{2} \tag{4}$$

$$d_1^2 + d_2^2 = \frac{1}{2} \tag{5}$$

$$a_1 \cdot b_1 + a_2 \cdot b_2 = 0 \tag{6}$$

$$c_1 \cdot d_1 + c_2 \cdot d_2 = 0 \tag{7}$$

where equations (2)$\sim$(5) represent the norm of the basis and equations (6),(7) represent the orthogonality conditions between the bases.

Note that by using the Lattice model expression [6], I_2 is the unit matrix of (2×2), and J_2 is the opposite-angle matrix (2×2) with 1 element. Let J_2 be the antisymmetric matrix (2×2) with element 1, and Equation (1) is expressed as follows

$$S_0 = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \tag{8}$$

$$S_1 = \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \tag{9}$$

$$T_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} S_0 & 0 \\ 0 & S_1 \end{pmatrix} \begin{pmatrix} I_2 & J_2 \\ I_2 & -J_2 \end{pmatrix} \tag{10}$$

In the same way, we can use

$$D = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{11}$$

then equations (2)$\sim$(7) are equivalent to the following equations.

$$T_4^t \cdot D \cdot T_4 = J_4 \tag{12}$$

On the other hand, by applying a rotation operation to the 4-dimensional unit matrix I_4, we can generate a basic symmetric matrix E_4 with a vertical axis of symmetry between columns 2 and 3 and a diagonal matrix of (2×2) symmetrically arranged, where the 1st and 2nd rows correspond to even-symmetric components and the 3rd and 4th rows to odd-symmetric components.

$$E_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{13}$$

By applying certain deformation operations to the basic symmetric matrix E_4, we can generate a symmetric all 4-dimensional orthonormal basis T_4.

2.2 Column manipulation to preserve symmetry of E_4

In order for T_4 to represent all orthonormal bases, we need to add a continuity deformation operation like rotation. Since each term of T_4 is a continuous real number, we first consider its finite group and then extend it to an infinite group. Therefore, we reveal the operation on the columns of the basic symmetric matrix E_4 that preserves its symmetry, orthogonality, and norm 1[14].

In general, swapping columns corresponds to a substitution operation on the coordinate axes of a 4-dimensional orthogonal space, and which constitutes a finite symmetry group (permutation group) $\mathcal{S}(4)$ of order 4[15][16]. The total number of substitutions, i.e. order, of $\mathcal{S}(4)$ is $4! = 24$. However, this does not necessarily mean that the conditions in equations (2)$\sim$(7) are satisfied. Substitution of columns with symmetry axes between columns 2 and 3 requires that the column numbers x_1 to x_4 satisfy the condition that the 4th-order symmetry equation f_4 is invariant[11].

$$f_4 = x_1 \cdot x_4 + x_2 \cdot x_3 \tag{14}$$

For example, simultaneous substitutions of column 1 (x_1) and column 2 (x_2), column 3 (x_3) and column 4 (x_4) will not affect the values of the symmetric formula.

It is clear that equations (2)∼(7) are satisfied in this case. As shown in Table 1, the number of substitutions satisfying the symmetry formula f_4 is $2!\ 2^2 = 8$, and constitutes a symmetry group.

In this paper, we call this the symmetric permutation group G_σ, where σ_0 to σ_3 are even permutation, σ_4 to σ_7 are odd permutation, σ_0 is identity permutation, and σ_4 and σ_5 are transposition.

There is also a relationship $\sigma_n^{-1} = \sigma_n$ $(n = 0, 1, \cdots, 5)$,$\sigma_6^{-1} = \sigma_7$,$\sigma_7^{-1} = \sigma_6$. The permutation σ_n

Table. 1 The symmetric permutation group G_σ formed by columns of matrix E_4

	column permutation	x_1	x_2	x_3	x_4	expression using transpositions(example)
even permutation	σ_0	1	2	3	4	identity permutation
	σ_1	2	1	4	3	(1,2)(3,4)
	σ_2	3	4	1	2	(1,3)(2,4)
	σ_3	4	3	2	1	(1,4)(2,3)
odd permutation	σ_4	1	3	2	4	transposition(2,3)
	σ_5	4	2	3	1	transposition(1,4)
	σ_6	2	4	1	3	(1,3)(2,4)(1,4)
	σ_7	3	1	4	2	(1,2)(3,4)(1,4)

can also be expressed using a (4×4) matrix T_{σ_n},$(n = 0, 1, \cdots, 7)$. In this case, $T_{\sigma_n}^{-1} = T_{\sigma_n}^{\,t}$ is an orthogonal matrix (real unitary matrix). Using this matrix T_{σ_n}, column substitution can be expressed as $E_4 \cdot T_{\sigma_n}$.

Since T_{σ_n} is multiplied from the right of the basic symmetric matrix E_4, these operations are called right basic deformations. Table 2 shows a group table summarizing the combinations (products) of these permutations. This shows that the product of permutations $(\sigma_m \sigma_n)$ is as noncommutative as the product of matrices $(T_{\sigma_m} T_{\sigma_n})$.

Table. 2 Multiplication table of the symmetric permutation group G_σ

	right \ left	even permutation				odd permutation			
		σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
even permutation	σ_0(1, 2, 3, 4)	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
	σ_1(2, 1, 4, 3)	σ_1	σ_0	σ_3	σ_2	σ_6	σ_7	σ_4	σ_5
	σ_2(3, 4, 1, 2)	σ_2	σ_3	σ_0	σ_1	σ_7	σ_6	σ_5	σ_4
	σ_3(4, 3, 2, 1)	σ_3	σ_2	σ_1	σ_0	σ_5	σ_4	σ_7	σ_6
odd permutation	σ_4(1, 3, 2, 4)	σ_4	σ_7	σ_6	σ_5	σ_0	σ_3	σ_2	σ_1
	σ_5(4, 2, 3, 1)	σ_5	σ_6	σ_7	σ_4	σ_3	σ_0	σ_1	σ_2
	σ_6(2, 4, 1, 3)	σ_6	σ_5	σ_4	σ_7	σ_1	σ_2	σ_3	σ_0
	σ_7(3, 1, 4, 2)	σ_7	σ_4	σ_5	σ_6	σ_2	σ_1	σ_0	σ_3

2.3 sign inversion operation to preserve the symmetry of E_4

In the previous section, we considered the permutation of columns that make the 4th-order symmetric formula f_4 invariant, but this condition is not only satisfied by substitution. For example, reversing the signs $(\pm)$ of x_2 in column 2 and x_3 in column 3 simultaneously does not change the form of the equation.

Here, we define the operations ρ_1, ρ_2 and ρ_3 that invert the symmetric column sign $(\pm)$ with respect to the operation $\rho_0(+,+,+,+)$ corresponding to the unit element as shown in Table 3. In this case, multiplication operation can be defined between ρ_0∼ρ_3, for example, the following relation is formed.

$$\rho_1 \cdot \rho_2 = \rho_2 \cdot \rho_1 = \rho_3 \tag{15}$$

As with the permutation in the previous section, it constitutes a symmetry group, which we will call the sign inversion group G_ρ. As with permutation, the element of the

Table. 3 Multiplication table of the sign inversion group G_ρ formed by columns of matrix E_4

right \ left	ρ_0	ρ_1	ρ_2	ρ_3
ρ_0 $(+,+,+,+)$	ρ_0	ρ_1	ρ_2	ρ_3
ρ_1 $(+,-,-,+)$	ρ_1	ρ_0	ρ_3	ρ_2
ρ_2 $(-,+,+,-)$	ρ_2	ρ_3	ρ_0	ρ_1
ρ_3 $(-,-,-,-)$	ρ_3	ρ_2	ρ_1	ρ_0

sign inversion group G_ρ can also be represented using an orthogonal matrix of (4×4). For example, ρ_1 is equal to the diagonal terms in columns 2 and 3 of the (4×4) unit matrix I_4 set to -1.

2.4 Symmetric permutation-sign inversion group G

The symmetric column substitution σ_0∼σ_7 constitutes the symmetric permutation group G_σ of order 8, and the symmetric sign-inverted operation ρ_0∼ρ_3 constitutes the sign inversion group G_ρ of order 4. They can all be represented by (4×4) orthogonal matrices with symmetric components, and We can define multiplication operation between them. As shown below, the multiplication of σ and ρ is noncommutative.

$$\sigma_1 \cdot \rho_1 = \rho_2 \cdot \sigma_1 \tag{16}$$

$$\sigma_1 \cdot \rho_2 = \rho_1 \cdot \sigma_1 \tag{17}$$

$$\sigma_2 \cdot \rho_1 = \rho_2 \cdot \sigma_2 \tag{18}$$

$$\sigma_2 \cdot \rho_2 = \rho_1 \cdot \sigma_2 \tag{19}$$

$$\sigma_6 \cdot \rho_1 = \rho_2 \cdot \sigma_6 \tag{20}$$

$$\sigma_6 \cdot \rho_2 = \rho_1 \cdot \sigma_6 \tag{21}$$

$$\sigma_7 \cdot \rho_1 = \rho_2 \cdot \sigma_7 \tag{22}$$

$$\sigma_7 \cdot \rho_2 = \rho_1 \cdot \sigma_7 \tag{23}$$

These combinations are called direct products $[G_\sigma \times G_\rho]$ and constitute a symmetry group of order 32 as shown in Table 4. In this paper, we call this the symmetric permutation and sign inversion group G.

Table. 4 Direct products of the symmetric permutation group G_σ and the sign inversion group G_ρ

symmetric permutation groupG_σ		sign inversion groupG_ρ			
		ρ_0 $(+,+,+,+)$	ρ_1 $(+,-,-,+)$	ρ_2 $(-,+,+,-)$	ρ_3 $(-,-,-,-)$
even permutation	σ_0(1, 2, 3, 4)	1, 2, 3, 4	1, −2, −3, 4	−1, 2, 3, −4	−1, −2, −3, −4
	σ_1(2, 1, 4, 3)	2, 1, 4, 3	2, −1, −4, 3	−2, 1, 4, −3	−2, −1, −4, −3
	σ_2(3, 4, 1, 2)	3, 4, 1, 2	3, −4, −1, 2	−3, 4, 1, −2	−3, −4, −1, −2
	σ_3(4, 3, 2, 1)	4, 3, 2, 1	4, −3, −2, 1	−4, 3, 2, −1	−4, −3, −2, −1
odd permutation	σ_4(1, 3, 2, 4)	1, 3, 2, 4	1, −3, −2, 4	−1, 3, 2, −4	−1, −3, −2, −4
	σ_5(4, 2, 3, 1)	4, 2, 3, 1	4, −2, −3, 1	−4, 2, 3, −1	−4, −2, −3, −1
	σ_6(2, 4, 1, 3)	2, 4, 1, 3	2, −4, −1, 3	−2, 4, 1, −3	−2, −4, −1, −3
	σ_7(3, 1, 4, 2)	3, 1, 4, 2	3, −1, −4, 2	−3, 1, 4, −2	−3, −1, −4, −2

2.5 Normal subgroup H of symmetric permutation and sign inversion group$G$

In order to clarify the structure of the symmetric permutation and sign inversion group G, we clarify its subgroups and their properties. There are multiple non-trivial true subgroups of G. The largest of these is the normal subgroup H_{16} with order 16 and index 2. Its elements are even-permutation σ_0 to σ_3 terms and are noncommutative.

Furthermore, there are 2 normal subgroups of order 8 and index 4 in G. The H_8 in it is commutative as shown in Table 5 and can be extended to a continuous rotation group.

Table. 5 Multiplication table of the normal subgroup H_8 in G

right \ left	$\sigma_0\rho_0$	$\sigma_0\rho_3$	$\sigma_1\rho_1$	$\sigma_1\rho_2$	$\sigma_2\rho_1$	$\sigma_2\rho_2$	$\sigma_3\rho_0$	$\sigma_3\rho_3$
$\sigma_0\rho_0$	$\sigma_0\rho_0$	$\sigma_0\rho_3$	$\sigma_1\rho_1$	$\sigma_1\rho_2$	$\sigma_2\rho_1$	$\sigma_2\rho_2$	$\sigma_3\rho_0$	$\sigma_3\rho_3$
$\sigma_0\rho_3$	$\sigma_0\rho_3$	$\sigma_0\rho_0$	$\sigma_1\rho_2$	$\sigma_1\rho_1$	$\sigma_2\rho_2$	$\sigma_2\rho_1$	$\sigma_3\rho_3$	$\sigma_3\rho_0$
$\sigma_1\rho_1$	$\sigma_1\rho_1$	$\sigma_1\rho_2$	$\sigma_0\rho_3$	$\sigma_0\rho_0$	$\sigma_3\rho_3$	$\sigma_3\rho_0$	$\sigma_2\rho_1$	$\sigma_2\rho_2$
$\sigma_1\rho_2$	$\sigma_1\rho_2$	$\sigma_1\rho_1$	$\sigma_0\rho_0$	$\sigma_0\rho_3$	$\sigma_3\rho_0$	$\sigma_3\rho_3$	$\sigma_2\rho_2$	$\sigma_2\rho_1$
$\sigma_2\rho_1$	$\sigma_2\rho_1$	$\sigma_2\rho_2$	$\sigma_3\rho_3$	$\sigma_3\rho_0$	$\sigma_0\rho_3$	$\sigma_0\rho_0$	$\sigma_1\rho_1$	$\sigma_1\rho_2$
$\sigma_2\rho_2$	$\sigma_2\rho_2$	$\sigma_2\rho_1$	$\sigma_3\rho_0$	$\sigma_3\rho_3$	$\sigma_0\rho_0$	$\sigma_0\rho_3$	$\sigma_1\rho_2$	$\sigma_1\rho_1$
$\sigma_3\rho_0$	$\sigma_3\rho_0$	$\sigma_3\rho_3$	$\sigma_2\rho_1$	$\sigma_2\rho_2$	$\sigma_1\rho_1$	$\sigma_1\rho_2$	$\sigma_0\rho_0$	$\sigma_0\rho_3$
$\sigma_3\rho_3$	$\sigma_3\rho_3$	$\sigma_3\rho_0$	$\sigma_2\rho_2$	$\sigma_2\rho_1$	$\sigma_1\rho_2$	$\sigma_1\rho_1$	$\sigma_0\rho_3$	$\sigma_0\rho_0$

2.6 Classification by residue class C_0-C_3 with normal subgroup H_8 as modulus

From the properties of the symmetry group, we can use H_8 to classify the elements of G into 4 residue classes C_0 to C_3 that have no elements in common with each other. The following residue classes C_0 to C_3 exist, all of which have order 8 as shown in Table 6,

$$G = C_0 \cup C_1 \cup C_2 \cup C_3 \tag{24}$$

$$C_j \cap C_k = \emptyset \qquad (j \neq k) \tag{25}$$

where C_0 to C_3 correspond to the residue classes (left and right residue classes) with the normal subgroup H_8 formed by G, and are expressed as follows.

$$C_0 = (\sigma_0 \cdot \rho_0) \cdot H_8 = H_8 \cdot (\sigma_0 \cdot \rho_0) = H_8 \tag{26}$$

$$C_1 = g_1 \cdot H_8 = H_8 \cdot g_1 \qquad (g_1 \in C_1) \tag{27}$$

$$C_2 = g_2 \cdot H_8 = H_8 \cdot g_2 \qquad (g_2 \in C_2) \tag{28}$$

$$C_3 = g_3 \cdot H_8 = H_8 \cdot g_3 \qquad (g_3 \in C_3) \tag{29}$$

Table. 6 Classification of G elements by the residue classes C_0, C_1, C_2, C_3

		σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
ρ_0	Residue class	$\sigma_0\rho_0$ C_0	$\sigma_1\rho_0$ C_1	$\sigma_2\rho_0$ C_1	$\sigma_3\rho_0$ C_0	$\sigma_4\rho_0$ C_2	$\sigma_5\rho_0$ C_2	$\sigma_6\rho_0$ C_3	$\sigma_7\rho_0$ C_3
ρ_1	Residue class	$\sigma_0\rho_1$ C_1	$\sigma_1\rho_1$ C_0	$\sigma_2\rho_1$ C_0	$\sigma_3\rho_1$ C_1	$\sigma_4\rho_1$ C_3	$\sigma_5\rho_1$ C_3	$\sigma_6\rho_1$ C_2	$\sigma_7\rho_1$ C_2
ρ_2	Residue class	$\sigma_0\rho_2$ C_1	$\sigma_1\rho_2$ C_0	$\sigma_2\rho_2$ C_0	$\sigma_3\rho_2$ C_1	$\sigma_4\rho_2$ C_3	$\sigma_5\rho_2$ C_3	$\sigma_6\rho_2$ C_2	$\sigma_7\rho_2$ C_2
ρ_3	Residue class	$\sigma_0\rho_3$ C_0	$\sigma_1\rho_3$ C_1	$\sigma_2\rho_3$ C_1	$\sigma_3\rho_3$ C_0	$\sigma_4\rho_3$ C_2	$\sigma_5\rho_3$ C_2	$\sigma_6\rho_3$ C_3	$\sigma_7\rho_3$ C_3

Next, we extend this finite normal subgroup H_8 to a symmetric subgroup of the 4th-order special orthogonal group $\mathcal{SO}(4)$, which is located in the topological group in the continuous group. Every element of H_8 of a normal subgroup can be represented by a pair of symmetric rotation operations on columns $R_1(\theta)$ and $R_2(\phi)$, respectively. In this paper, we call this a symmetric rotation pair.

$$R_1(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{30}$$

$$R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix} \tag{31}$$

The continuous real variable θ, ϕ corresponds to the angle of rotation, and the product of these rotations $R_1(\theta) \cdot R_2(\phi)$ is commutative. By using this symmetric rotation pair $R_1(\theta)$, $R_2(\phi)$, we define 4 models that extend the residue class C_0 $\sim C_3$. In each of these models, by giving the rotation angles of 0, $\pm\frac{\pi}{2}$, π shown in Tables 7 and 8 we can represent all the elements of G.

2.7 Basic rotation model using symmetric rotation pairs

By considering θ, ϕ rotation parameters of symmetric rotation pairs R_1 and R_2 as continuous quantities, we define 4 basic rotation models I∼IV corresponding to the residue class $C_0 \sim C_3$.

2.7.1 Basic rotation model I ($H_8 = C_0$)

The reference rotation model, represented by E_4 $R_1(\theta)$ $R_2(\phi)$ with continuous rotation parameters.

Table. 7 An expression of the symmetric permutation and sign inversion group G using a pair of symmetric rotations (left residue classes)

rotation angle		$C_0 = H_8$	left residue class C_1	left residue class C_2	left residue class C_3
θ	ϕ	$R_1(\theta)R_2(\phi)$	$\rho_1 R_1(\theta)R_2(\phi)$	$\sigma_4 R_1(\theta)R_2(\phi)$	$\sigma_4 \rho_1 R_1(\theta)R_2(\phi)$
0	0	$\sigma_0\rho_0$	$\sigma_0\rho_1$	$\sigma_4\rho_0$	$\sigma_4\rho_1$
	$\frac{\pi}{2}$	$\sigma_2\rho_1$	$\sigma_2\rho_3$	$\sigma_7\rho_1$	$\sigma_7\rho_3$
	$-\frac{\pi}{2}$	$\sigma_2\rho_2$	$\sigma_2\rho_0$	$\sigma_7\rho_2$	$\sigma_7\rho_0$
	π	$\sigma_0\rho_3$	$\sigma_0\rho_2$	$\sigma_4\rho_3$	$\sigma_4\rho_2$
$\frac{\pi}{2}$	0	$\sigma_1\rho_1$	$\sigma_1\rho_3$	$\sigma_6\rho_1$	$\sigma_6\rho_3$
	$\frac{\pi}{2}$	$\sigma_3\rho_3$	$\sigma_3\rho_2$	$\sigma_5\rho_3$	$\sigma_5\rho_2$
	$-\frac{\pi}{2}$	$\sigma_3\rho_0$	$\sigma_3\rho_1$	$\sigma_5\rho_0$	$\sigma_5\rho_1$
	π	$\sigma_1\rho_2$	$\sigma_1\rho_0$	$\sigma_6\rho_2$	$\sigma_6\rho_0$
$-\frac{\pi}{2}$	0	$\sigma_1\rho_2$	$\sigma_1\rho_0$	$\sigma_6\rho_2$	$\sigma_6\rho_0$
	$\frac{\pi}{2}$	$\sigma_3\rho_0$	$\sigma_3\rho_1$	$\sigma_5\rho_0$	$\sigma_5\rho_1$
	$-\frac{\pi}{2}$	$\sigma_3\rho_3$	$\sigma_3\rho_2$	$\sigma_5\rho_3$	$\sigma_5\rho_2$
	π	$\sigma_1\rho_1$	$\sigma_1\rho_3$	$\sigma_6\rho_1$	$\sigma_6\rho_3$
π	0	$\sigma_0\rho_3$	$\sigma_0\rho_2$	$\sigma_4\rho_3$	$\sigma_4\rho_2$
	$\frac{\pi}{2}$	$\sigma_2\rho_2$	$\sigma_2\rho_0$	$\sigma_7\rho_2$	$\sigma_7\rho_0$
	$-\frac{\pi}{2}$	$\sigma_2\rho_1$	$\sigma_2\rho_3$	$\sigma_7\rho_1$	$\sigma_7\rho_3$
	π	$\sigma_0\rho_0$	$\sigma_0\rho_1$	$\sigma_4\rho_0$	$\sigma_4\rho_1$

Table. 8 An expression of the symmetric permutation and sign inversion group G using a pair of symmetric rotations (right residue classes)

rotation angle		$C_0 = H_8$	right residue class C_1	right residue class C_2	right residue class C_3
θ	ϕ	$R_1(\theta)R_2(\phi)$	$R_1(\theta)R_2(\phi)\rho_1$	$R_1(\theta)R_2(\phi)\sigma_4$	$R_1(\theta)R_2(\phi)\sigma_4\rho_1$
0	0	$\sigma_0\rho_0$	$\sigma_0\rho_1$	$\sigma_4\rho_0$	$\sigma_4\rho_1$
	$\frac{\pi}{2}$	$\sigma_2\rho_1$	$\sigma_2\rho_0$	$\sigma_6\rho_1$	$\sigma_6\rho_0$
	$-\frac{\pi}{2}$	$\sigma_2\rho_2$	$\sigma_2\rho_3$	$\sigma_6\rho_2$	$\sigma_6\rho_3$
	π	$\sigma_0\rho_3$	$\sigma_0\rho_2$	$\sigma_4\rho_3$	$\sigma_4\rho_2$
$\frac{\pi}{2}$	0	$\sigma_1\rho_1$	$\sigma_1\rho_0$	$\sigma_7\rho_1$	$\sigma_7\rho_0$
	$\frac{\pi}{2}$	$\sigma_3\rho_3$	$\sigma_3\rho_2$	$\sigma_5\rho_3$	$\sigma_5\rho_2$
	$-\frac{\pi}{2}$	$\sigma_3\rho_0$	$\sigma_3\rho_1$	$\sigma_5\rho_0$	$\sigma_5\rho_1$
	π	$\sigma_1\rho_2$	$\sigma_1\rho_3$	$\sigma_7\rho_2$	$\sigma_7\rho_3$
$-\frac{\pi}{2}$	0	$\sigma_1\rho_2$	$\sigma_1\rho_3$	$\sigma_7\rho_2$	$\sigma_7\rho_3$
	$\frac{\pi}{2}$	$\sigma_3\rho_0$	$\sigma_3\rho_1$	$\sigma_5\rho_0$	$\sigma_5\rho_1$
	$-\frac{\pi}{2}$	$\sigma_3\rho_3$	$\sigma_3\rho_2$	$\sigma_5\rho_3$	$\sigma_5\rho_2$
	π	$\sigma_1\rho_1$	$\sigma_1\rho_0$	$\sigma_7\rho_1$	$\sigma_7\rho_0$
π	0	$\sigma_0\rho_3$	$\sigma_0\rho_2$	$\sigma_4\rho_3$	$\sigma_4\rho_2$
	$\frac{\pi}{2}$	$\sigma_2\rho_2$	$\sigma_2\rho_3$	$\sigma_6\rho_2$	$\sigma_6\rho_3$
	$-\frac{\pi}{2}$	$\sigma_2\rho_1$	$\sigma_2\rho_0$	$\sigma_6\rho_1$	$\sigma_6\rho_0$
	π	$\sigma_0\rho_0$	$\sigma_0\rho_1$	$\sigma_4\rho_0$	$\sigma_4\rho_1$

2.7.2 Basic rotation model Ⅱ (C_1)

There are many variations of the basic rotation model corresponding to the residue class C_1. For example, using the left residue class of $\rho_1 \in C_1$, the basic rotation model is represented by $E_4\ \rho_1\ R_1(\theta)\ R_2(\phi)$.

2.7.3 Basic rotation model Ⅲ (C_2)

The basic rotation model corresponding to the residue class C_2 is represented by $E_4\ \sigma_4\ R_1(\theta)\ R_2(\phi)$, using the left residue class of $\sigma_4 \in C_2$.

2.7.4 Basic rotation model Ⅳ (C_3)

The basic rotation model corresponding to the residue class C_3 is represented by $E_4\ \sigma_4\ \rho_1\ R_1(\theta)\ R_2(\phi)$, using the left residue class of $\sigma_4\ \rho_1 \in\ C_3$. Note that we can also express the basic rotation model Ⅱ to Ⅳ by using the right residue class in Table 8, where the basic rotation model (Ⅰ) and (Ⅱ) are symmetric subgroups of the special orthogonal group $\mathcal{SO}(4)$, and the basic rotation model (Ⅰ)∼(Ⅳ) correspond to symmetric subgroups of the orthogonal group $\mathcal{O}(4)$[15].

2.8 Equivalent transformation of basic rotation models

The components of the basic rotation model, $R_1(\theta)$, $R_2(\phi)$, ρ_1, ρ_2, and σ_4 belong to the right basic deformation for E_4. For these operations, the equivalent transformation rules shown in Figure 1 are satisfied. By applying this rule one after the other, it is possible to represent all Ⅰ∼Ⅳ in the basic rotation model using only the left basic deformation of E_4.

3 Rotation models of 4-dimensional generalized LOT

This chapter describes methods for constructing 4 dimension generalized LOT rotation models.

3.1 Construction method of 4-dimensional LOT(4×8)

In the previous section, we performed 4 operations on the 4 columns of the basic symmetric matrix E_4, corresponding to right basic deformations. It was shown that this can generate all orthonormal bases satisfying the 3 conditions of symmetry, orthogonality, and norm 1. The length of the orthogonal basis is extended to 8 columns of 2 blocks by applying the same operation to the above basis with a shift of half a block, i.e., 2 columns, and all the bases of LOT(4×8) satisfying the above 3 conditions are generated. For each of the 4 types of operations (Ⅰ∼Ⅳ) in the 1st stage shown in the previous section, the rotation operations in the 2nd

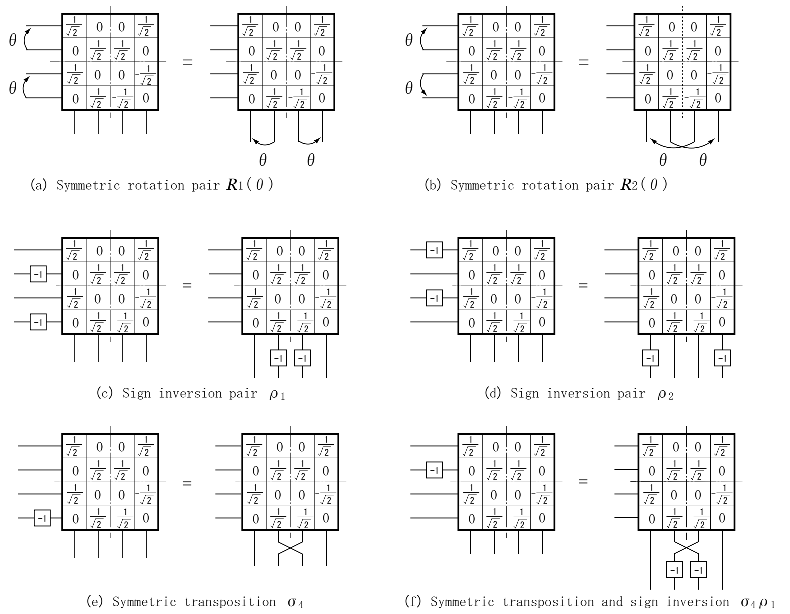


Fig. 1 Equivalent transformation rules on the basic symmetric matrix E_4

stage (I∼IV) are added, so the rotation model is expanded to 16 types of (I−I)∼(IV−IV).

Figure 2(a) shows an example of model (I−I) and Figure 2(b) shows an example of model (Ⅲ−Ⅲ). In this example, the right residue class representation is used, however it can also be expressed in terms of the left residue class.

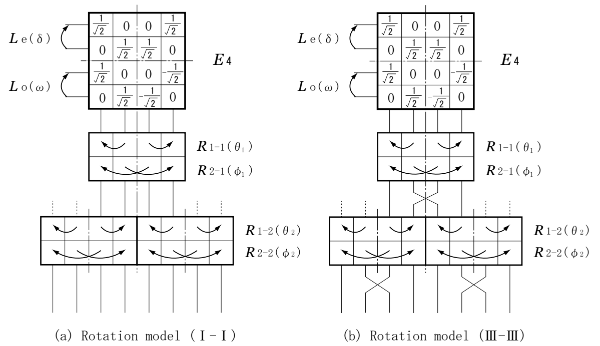


Fig. 2 Rotation models of LOT(4×8)(I−I) and (Ⅲ−Ⅲ)

3.2 Equivalent transformation rules between stages

By using the property that the product of R_1 and R_2 of the rotation operation is commutative, for example, $R_{1-2}(\theta_2)$ of the 2nd stage in Figure 2(a) can be moved to the 1st stage and absorbed into $R_{1-1}(\theta_1)$. In this case, since the direction of their rotation is reversed, we can omit the parameter θ_2 by defining the angle used in $R_{1-1}(\theta_1 - \theta_2)$ as a new angle θ_1. In the case of Figure 2(b) on the other hand, the position of R_1 and R_2 can be moved over σ_4, which corresponds to the cross section of each stage.

For example, $R_{1-2}(\theta_2)$ in the 2nd stage becomes an equivalent operation to R_2 when it exchanges positions with σ_4 and is absorbed by $R_{2-1}(\phi_1)$ in the 1st stage. This indicates that θ_2 is a redundant parameter. As shown in the previous chapter, all the operations of the 1st stage can be represented by the left basic deformation. Finally, the minimum number of parameters for LOT(4×8) is ϕ_2 in the 2nd stage and δ and ω in the left basic transformation[13]. Thus, there are redundant operations between stages, and unnecessary rotation parameters can be organized and integrated. In this case, the following equivalence transformation rules are formed.

$$R_1(\theta)\rho_1 = \rho_1 R_1(-\theta) \tag{32}$$

$$R_2(\theta)\rho_1 = \rho_1 R_2(-\theta) \tag{33}$$

$$R_1(\theta)\sigma_4 = \sigma_4 R_2(\theta) \tag{34}$$

$$R_2(\theta)\sigma_4 = \sigma_4 R_1(\theta) \tag{35}$$

These rules are represented graphically in Figure 3.

3.3 Equivalent transformation rules for coding gain

The coding gain is widely used as a measure to evaluate the coding efficiency of orthogonal transforms such as LOT. As is clear from its definition, the value of the coding gain does not change when operations such as substituting the basis of the orthogonal transform, sign inversion of ±, or mirroring are performed[13]. By integrating the equivalent transformation rule for the basic symmetric matrix E_4

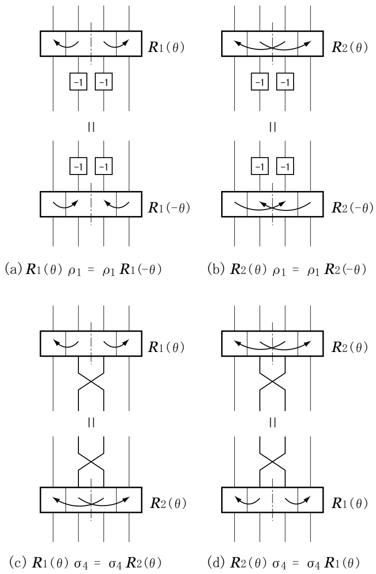


Fig. 3 Equivalent transformation rules between stages

shown in Figure 1 and the inter-stage equivalent transformation rule shown in Equations (32)∼(35), we can derive an equivalent transformation rule that makes the maximum coding gain invariant.

For example, moving ρ_1 in the 2nd stage to the 1st stage is equivalent to ρ_2, which is finally aggregated to the left basic deformation. At the same time, all operations in the 1st stage are absorbed by the left basic deformation, so that the maximum values of their coding gains are equal in the rotation model I and II. This relationship holds for the rotation models III and IV, leading to the following equivalent transformation rules.

$$\mathrm{I} \Leftrightarrow \mathrm{II} \tag{36}$$

$$\mathrm{III} \Leftrightarrow \mathrm{IV} \tag{37}$$

3.4 Classification of LOT rotation models (4×8)

As shown in the previous section, there are 16 rotation models for LOT(4×8) in total. However, using the equivalent transformation rules (32)∼(35) for the coding gains, they are effectively classified into 2 groups. Basically, it is sufficient to search for the parameters that maximize the encoding gain for the 2 rotation models that represent this group. In fact, for all (4×8) models with stage I, simulations were performed to find the optimal solution of the rotation parameter that maximizes the coding gain.

Table 9 shows the values when the autocorrelation coefficient ρ of the coding gain is set to 0.95. The coding gain was 7.960 (dB) for the 8 models for which the 2nd stage was I and II, and 7.782 (dB) for the 8 models for which III and IV. That is, the values were finally classified into 2 groups independent of the 1st stage of operation. Note that when the 1st stage is II, III, IV, the values of the rotation angles δ and ω of L_e and L_o may be inverted. However, the coding gain and the value of the rotation angle θ_2 of R_{2-2} remain unchanged .

Table. 9 Optimum parameters of LOT(4×8) (2 groups)

Construction	Optimum parameters			Coding gain (max)
2nd stage	R_{2-2}	L_e	L_o	(1st stage : I)
I	α_1	α_2	α_3	7.960(dB)
II	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	
III	β_1	β_2	β_3	7.782(dB)
IV	$-\beta_1$	$-\beta_2$	$-\beta_3$	

$\alpha_1 = 0.054\pi$, $\alpha_2 = 0.196\pi$, $\alpha_3 = -0.202\pi$
$\beta_1 = 0.181\pi$, $\beta_2 = 0.071\pi$, $\beta_3 = 0.132\pi$

3.5 Construction method of GenLOT rotation models(4×12)

As shown in Figure 4, the base length is extended from 8 to 12 by adding operations such as the rotation of the 3rd stage (I∼IV) to the rotational model of LOT(4×8). This operation allows us to construct all bases of the generalized LOT(4×12). The total number of combinations is 64 (I−I−I)∼(IV−IV−IV).

Figure 4 shows an example of model (III−III−III). Note that this expression uses the right residue class. In the previous paper on rotation models[13], models using II,III, IV after the 2nd stage have not been considered.

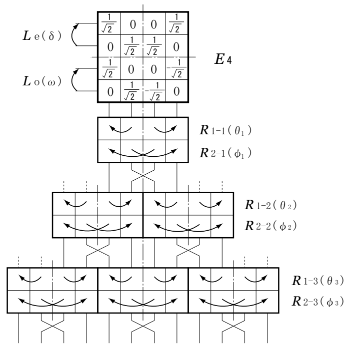


Fig. 4 Rotation model of GenLOT(4×12) (III−III−III)

3.6 Equivalent Conversion Rules of Coding Gain for GenLOT(4×12)

The generalized LOT rotation model (III−III−III) in Figure 4 can be equivalently transformed to the form (I−III−I) by the following procedure.

1. Cross section of 3rd stage (σ_4) $\rightarrow$ 2nd stage
2. 2nd stage R_{2-2} $\rightarrow$ 3rd stage
3. Cross section of 1st stage (σ_4) $\rightarrow$ 2nd stage
4. 2nd stage R_{1-2} $\rightarrow$ 1st stage
5. 3rd stage R_{2-3} $\rightarrow$ 2nd stage

The process of these operations is shown in Figure 5.

Here, R_{1-2} in the 2nd stage in Figure 5(a) is integrated into R_{2-1} in the 1st stage and R_{1-3} in the 3rd stage into R_{2-2} in the 2nd stage. Thus, the final 2nd stage is R_{2-2} and the 3rd stage is only R_{2-3}. Also, R_{1-3} in the 3rd stage of Figure 5(c) is integrated into R_{2-2}.

Similarly, the rotation model (I−Ⅲ−Ⅲ) of the generalized LOT can be equivalently transformed into the form (Ⅲ−ⅡI−I) as follows.

1. Cross section of 3rd stage (σ_4) $\rightarrow$ 2nd stage
2. 2nd stage R_{2-2} $\rightarrow$ 3rd stage
3. Cross section of 2nd stage (σ_4) $\rightarrow$ 1st stage
4. 2nd stage R_{1-2} $\rightarrow$ 1st stage
5. 3rd stage R_{2-3} $\rightarrow$ 2nd stage

The process of these operations is shown in Figure 6. Note that adding the transposition σ_4 to the lower part of the 3rd stage in Figures 5(a) and 5(c) yields (c) and (a) in Figure 6, respectively. As a result, the following equivalence transformation rules for the coding gain are derived.

$$\text{Ⅲ} - \text{Ⅲ} - \text{Ⅲ} \;\Leftrightarrow\; \text{I} - \text{Ⅲ} - \text{I} \tag{38}$$

$$\text{I} - \text{Ⅲ} - \text{Ⅲ} \;\Leftrightarrow\; \text{Ⅲ} - \text{Ⅲ} - \text{I} \tag{39}$$

Note that the generalized LOT (4×12) is finally integrated into 4 parameters $R_{2-2}(\phi_2)$, $R_{2-3}(\phi_3)$, $L_e(\delta)$ and $L_o(\omega)$. Next, for the generalized LOT (4×12), the optimal solution for the rotation parameter that maximizes the encoding gain was obtained by computer simulation.

The results are shown in Table 10 for the case where the 1st stage is (I). From this, we confirmed that the coding gains are classified into 3 groups: ①8.214 (dB), ②8.067 (dB), and ③8.014 (dB). Note that all operations in the 1st stage are absorbed by the left basic deformation, so they are optional. All models whose 2nd stage is Ⅲ, or Ⅳ are merged into one group ③ by the rules (32)∼(35).

3.7 Construction method of GenLOT(4×16) and its classification

As shown in Figure 7, the base length is extended from 12 to 16 by adding a 4th stage (I Ⅳ) rotation operation to

Table. 10 Optimum parameters of GenLOT(4×12) (3 groups)

stage		Optimum parameters				Coding gain(max) (1st stage:I)
2nd	3rd	R_{2-2}	R_{2-3}	L_e	L_o	
I	I	α_1	α_2	α_3	α_4	8.214(dB)
	II	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$\alpha_1 = 0.074\pi$, $\alpha_1 = -0.121\pi$
II	I	$-\alpha_1$	α_2	$-\alpha_3$	$-\alpha_4$	$\alpha_3 = 0.060\pi$, $\alpha_4 = -0.074\pi$
	II	α_1	$-\alpha_2$	α_3	α_4	
I	III	β_1	β_2	β_3	β_4	8.067(dB)
	IV	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$	$\beta_1 = 0.067\pi$,$\beta_2 = -0.031\pi$
II	III	$-\beta_1$	β_2	$-\beta_3$	$-\beta_4$	$\beta_3 = 0.156\pi$,$\beta_4 = 0.128\pi$
	IV	β_1	$-\beta_2$	β_3	β_4	
III	I	γ_1	γ_2	γ_3	γ_4	8.014(dB)
	II	$-\gamma_1$	$-\gamma_2$	$-\gamma_3$	$-\gamma_4$	$\gamma_1 = 0.016\pi$, $\gamma_2 = -0.151\pi$
	III	$-\gamma_2$	$-\gamma_1$	γ_3	$-\gamma_4$	$\gamma_3 = 0.089\pi$, $\gamma_4 = 0.034\pi$
	IV	γ_2	γ_1	$-\gamma_3$	γ_4	
IV	I	$-\gamma_1$	γ_2	$-\gamma_3$	$-\gamma_4$	
	II	γ_1	$-\gamma_2$	γ_3	γ_4	
	III	γ_2	$-\gamma_1$	$-\gamma_3$	γ_4	
	IV	$-\gamma_2$	γ_1	γ_3	$-\gamma_4$	

the (4×12) rotation model. This construction allows us to represent all bases of the generalized LOT(4×16).

The number of combinations is 256 (I−I−I−I)∼(Ⅳ−Ⅳ−IV−Ⅳ). The figure shows the model of (Ⅲ−Ⅲ−I−Ⅲ), which is also expressed in terms of right residue class.

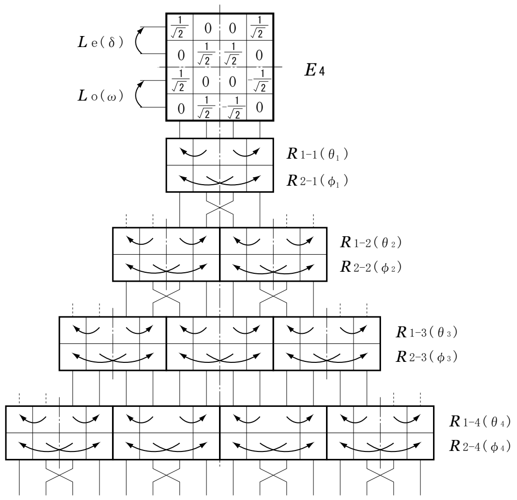
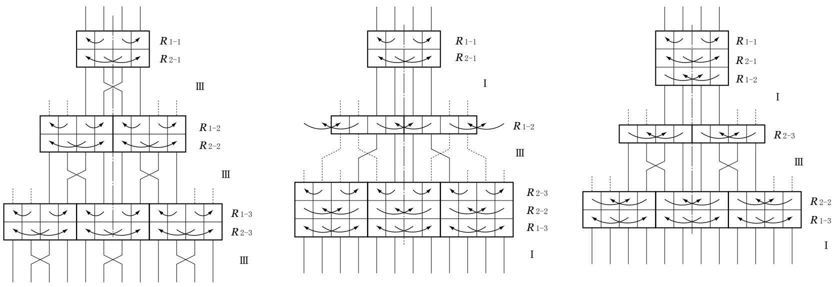


Fig. 7 Rotation model of GenLOT(4×16) (Ⅲ-Ⅲ-I-Ⅲ)

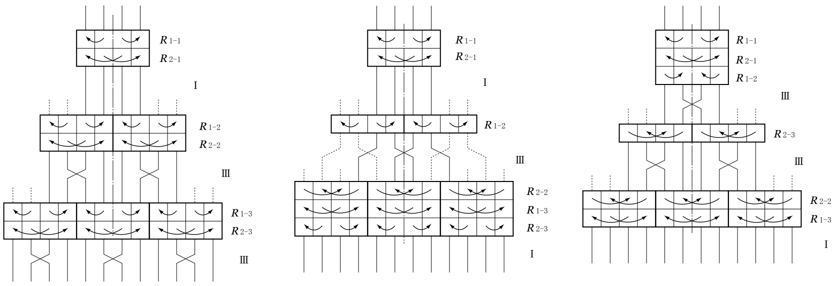
As in the previous section, the equivalent transformation rules in (32)∼(35) can be applied to the (4×16) rotation model. Here, since (32) and (33) can be applied to 3 consecutive stages, they are finally classified into 4 groups representing the models ①(I−I−I−I), ②(I−I−I−Ⅲ), ③(I−I−ⅡI−I) and ④(I−Ⅲ−I−I). (4×16) for the generalized LOT, the parameters that maximize the coding gain are obtained as shown in Table 11. Here, the 1st stage is optional, so the basic (I) is chosen. By the operation of equivalence transformation, the rotation parameters are integrated into five parameters $R_{2-2}(\theta_2)$,$R_{2-3}(\theta_3)$,$R_{2-4}(\theta_4)$,$L_e(\delta)$, and $L_o(\omega)$.

So far, we have organized methods to classify the rotation models of the generalized LOT with (4×8), (4×12),



(a) Ⅲ-Ⅲ-Ⅲ (b) Intermediate type (c) Ⅰ-Ⅲ-Ⅰ

Fig. 5 Equivalent transformation rules on coding gain of 4 dimensional GenLOT (Ⅲ−Ⅲ−Ⅲ $\Leftrightarrow$ Ⅰ−Ⅲ−Ⅰ)



(a) Ⅰ-Ⅲ-Ⅲ (b) Intermediate type (c) Ⅲ-Ⅲ-Ⅰ

Fig. 6 Equivalent transformation rules on coding gain of 4 dimensional GenLOT (Ⅰ−Ⅲ−Ⅲ $\Leftrightarrow$ Ⅲ−Ⅲ−Ⅰ)

and (4×16) into groups with equal maximum coding gains. When the number of stages is set to 5 or more, the equivalent transformation rules (32)∼(35) can be applied to classify them into the same number of groups as the number of stages. The optimal solution for the rotation parameters is one of the solutions, to which permutations and sign inversions are added to one of the solutions.

4 Construction method of 6-dimensional orthonormal basis

4.1 Basis symmetric matrix E_6 of orthonormal basis

For a 6-dimensional LOT, the (6×6) basic symmetric matrix E_6 is used as follows.

$$E_6 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \qquad (40)$$

Rows 1-3 of this matrix correspond to even-symmetric components, and rows 4-6 to odd-symmetric component. Next, we consider finite permutation operations that preserve the 3 conditions of symmetry, orthogonality, and norm 1 of E_6

4.2 Symmetric permutation group G_σ formed by columns of E_6

The order of the permutation group consisting of 6 elements is $6! = 720$, but not all of them preserve symmetry. The symmetry-preserving substitution group of E_6 for the symmetry axis between columns 3 and 4 satisfies the 6-dimensional symmetry equation shown below.

$$f_6 = x_1 \cdot x_6 + x_2 \cdot x_5 + x_3 \cdot x_4 \qquad (41)$$

The order of this symmetric permutation group G_σ is 48, as shown in Table 12, and even and odd permutations are equal in number, 24.

Note that there are 6 symmetric transposition pairs of even permutations with $\sigma_1, \cdots, \sigma_6$, and 3 odd transposition with $\sigma_a, \sigma_b, and \sigma_c$. In particular, symmetric transposition pairs correspond to the symmetric rotation pairs shown in the next section, and each stage has the same number of rotation parameters, 6.

4.3 The sign inversion group G_ρ for the columns of E_6

As in the 4-dimensional case, the sign inversion group G_ρ for the E_6 columns yields its order to be $2^3 = 8$. The group table is shown in Table 13. Note that G_σ is noncommutative, but this G_ρ is commutative.

Table. 11 Optimum parameters of GenLOT(4×16) (4 groups)

stage			optimum parameters					Coding gain(max)
2nd	3rd	4th	R_{2-2}	R_{2-3}	R_{2-4}	L_e	L_o	(1st stageI)
I	I	I	α_1	α_2	α_3	α_4	α_5	
		II	$-\alpha_1$	$-\alpha_2$	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	8.359(dB)
	II	I	$-\alpha_1$	$-\alpha_2$	α_3	$-\alpha_4$	$-\alpha_5$	
		II	α_1	α_2	$-\alpha_3$	α_4	α_5	$\alpha_1 = -0.104\pi$
II	I	I	$-\alpha_1$	α_2	α_3	$-\alpha_4$	$-\alpha_5$	$\alpha_2 = 0.129\pi$
		II	α_1	$-\alpha_2$	$-\alpha_3$	α_4	α_5	$\alpha_3 = 0.163\pi$
	II	I	α_1	$-\alpha_2$	α_3	α_4	α_5	$\alpha_4 = -0.183\pi$
		II	$-\alpha_1$	α_2	$-\alpha_3$	$-\alpha_4$	$-\alpha_5$	$\alpha_5 = 0.186\pi$
I	I	III	β_1	β_2	β_3	β_4	β_5	
		IV	$-\beta_1$	$-\beta_2$	$-\beta_3$	$-\beta_4$	$-\beta_5$	8.223(dB)
	II	III	$-\beta_1$	$-\beta_2$	β_3	$-\beta_4$	$-\beta_5$	
		IV	β_1	β_2	$-\beta_3$	β_4	β_5	$\beta_1 = -0.134\pi$
II	I	III	$-\beta_1$	β_2	β_3	$-\beta_4$	$-\beta_5$	$\beta_2 = 0.103\pi$
		IV	β_1	$-\beta_2$	$-\beta_3$	β_4	β_5	$\beta_3 = 0.325\pi$
	II	III	β_1	$-\beta_2$	β_3	β_4	β_5	$\beta_4 = 0.160\pi$
		IV	$-\beta_1$	β_2	$-\beta_3$	$-\beta_4$	$-\beta_5$	$\beta_5 = 0.040\pi$
I	III	I	γ_1	γ_2	γ_3	γ_4	γ_5	
		II	$-\gamma_1$	$-\gamma_2$	$-\gamma_3$	$-\gamma_4$	$-\gamma_5$	
	IV	I	$-\gamma_1$	$-\gamma_2$	γ_3	$-\gamma_4$	$-\gamma_5$	
		II	γ_1	γ_2	$-\gamma_3$	γ_4	γ_5	
II	III	I	$-\gamma_1$	γ_2	γ_3	$-\gamma_4$	$-\gamma_5$	
		II	γ_1	$-\gamma_2$	$-\gamma_3$	γ_4	γ_5	
	IV	I	γ_1	$-\gamma_2$	γ_3	γ_4	γ_5	
		II	$-\gamma_1$	γ_2	$-\gamma_3$	$-\gamma_4$	$-\gamma_5$	
III	I	III	γ_3	$-\gamma_1$	$-\gamma_2$	γ_4	$-\gamma_5$	
		IV	$-\gamma_3$	γ_1	γ_2	$-\gamma_4$	γ_5	8.220(dB)
	II	III	$-\gamma_3$	γ_1	$-\gamma_2$	$-\gamma_4$	γ_5	
		IV	γ_3	$-\gamma_1$	γ_2	γ_4	$-\gamma_5$	
	III	III	γ_1	$-\gamma_3$	$-\gamma_2$	γ_4	γ_5	
		IV	$-\gamma_1$	γ_3	γ_2	$-\gamma_4$	$-\gamma_5$	
	IV	III	$-\gamma_1$	γ_3	$-\gamma_2$	$-\gamma_4$	$-\gamma_5$	
		IV	γ_1	$-\gamma_3$	γ_2	γ_4	γ_5	$\gamma_1 = -0.057\pi$
IV	I	III	$-\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_4$	γ_5	$\gamma_2 = -0.379\pi$
		IV	γ_3	γ_1	γ_2	γ_4	$-\gamma_5$	$\gamma_3 = 0.104\pi$
	II	III	γ_3	γ_1	$-\gamma_2$	γ_4	$-\gamma_5$	$\gamma_4 = -0.179\pi$
		IV	$-\gamma_3$	$-\gamma_1$	γ_2	$-\gamma_4$	γ_5	$\gamma_5 = 0.110\pi$
	III	III	$-\gamma_1$	$-\gamma_3$	$-\gamma_2$	$-\gamma_4$	$-\gamma_5$	
		IV	γ_1	γ_3	γ_2	γ_4	γ_5	
	IV	III	γ_1	γ_3	$-\gamma_2$	γ_4	γ_5	
		IV	$-\gamma_1$	$-\gamma_3$	γ_2	$-\gamma_4$	$-\gamma_5$	
I	III	III	$-\delta_2$	$-\delta_3$	δ_1	δ_4	$-\delta_5$	
		IV	δ_2	δ_3	$-\delta_1$	$-\delta_4$	δ_5	
	IV	III	δ_2	δ_3	δ_1	$-\delta_4$	δ_5	
		IV	$-\delta_2$	$-\delta_3$	$-\delta_1$	δ_4	$-\delta_5$	
II	III	III	δ_2	$-\delta_3$	δ_1	$-\delta_4$	δ_5	
		IV	$-\delta_2$	δ_3	$-\delta_1$	δ_4	$-\delta_5$	
	IV	III	$-\delta_2$	δ_3	δ_1	δ_4	$-\delta_5$	
		IV	δ_2	$-\delta_3$	$-\delta_1$	$-\delta_4$	δ_5	
III	I	I	δ_1	δ_2	δ_3	δ_4	δ_5	8.277(dB)
		II	$-\delta_1$	$-\delta_2$	$-\delta_3$	$-\delta_4$	$-\delta_5$	
	II	I	$-\delta_1$	$-\delta_2$	δ_3	$-\delta_4$	$-\delta_5$	
		II	δ_1	δ_2	$-\delta_3$	δ_4	δ_5	
	III	I	$-\delta_2$	$-\delta_1$	δ_3	δ_4	$-\delta_5$	
		II	δ_2	δ_1	$-\delta_3$	$-\delta_4$	δ_5	
	IV	I	δ_2	δ_1	δ_3	$-\delta_4$	δ_5	$\delta_1 = 0.255\pi$
		II	$-\delta_2$	$-\delta_1$	$-\delta_3$	δ_4	$-\delta_5$	$\delta_2 = 0.080\pi$
IV	I	I	$-\delta_1$	δ_2	δ_3	$-\delta_4$	$-\delta_5$	$\delta_3 = 0.214\pi$
		II	δ_1	$-\delta_2$	$-\delta_3$	δ_4	δ_5	$\delta_4 = -0.141\pi$
	II	I	δ_1	$-\delta_2$	δ_3	δ_4	δ_5	$\delta_5 = -0.132\pi$
		II	$-\delta_1$	δ_2	$-\delta_3$	$-\delta_4$	$-\delta_5$	
	III	I	δ_2	$-\delta_1$	δ_3	$-\delta_4$	δ_5	
		II	$-\delta_2$	δ_1	$-\delta_3$	δ_4	$-\delta_5$	
	IV	I	$-\delta_2$	δ_1	δ_3	δ_4	$-\delta_5$	
		II	δ_2	$-\delta_1$	$-\delta_3$	$-\delta_4$	δ_5	

Table. 12 The symmetric permutation group G_σ formed by columns of matrix E_6

even permutation							odd permutation						
1	2	3	4	5	6	identity σ_0	1	2	4	3	5	6	trans- σ_a
1	3	2	5	4	6	σ_1	1	5	3	4	2	6	position σ_b
3	2	1	6	5	4	symme- σ_2	6	2	3	4	5	1	σ_c
2	1	3	4	6	5	tric σ_3	1	3	5	2	4	6	
1	4	5	2	3	6	pair of σ_4	1	4	2	5	3	6	
4	2	6	1	5	3	trans- σ_5	2	1	4	3	6	5	
5	6	3	4	1	2	position σ_6	2	3	6	1	4	5	
2	3	1	6	4	5		2	4	1	6	3	5	
3	1	2	5	6	4		2	6	3	4	1	5	
1	5	4	3	2	6		3	1	5	2	6	4	
2	4	6	1	3	5		3	2	6	1	5	4	
2	6	4	3	1	5		3	5	1	6	2	4	
3	5	6	1	2	4		3	6	2	5	1	4	
3	6	5	2	1	4		4	1	2	5	6	3	
4	1	5	2	6	3		4	2	1	6	5	3	
4	5	1	6	2	3		4	5	6	1	2	3	
4	6	2	5	1	3		4	6	5	2	1	3	
5	1	4	3	6	2		5	1	3	4	6	2	
5	3	6	1	4	2		5	3	1	6	4	2	
5	4	1	6	3	2		5	4	6	1	3	2	
6	2	4	3	5	1		5	6	4	3	1	2	
6	3	5	2	4	1		6	3	2	5	4	1	
6	4	2	5	3	1		6	4	5	2	3	1	
6	5	3	4	2	1		6	5	4	3	2	1	

Table. 13 Multiplication table of the sign inversion group G_ρ formed by columns of matrix E_6

right \ left	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
ρ_0 $(+,+,+,+,+,+)$	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
ρ_1 $(+,+,-,-,+,+)$	ρ_1	ρ_0	ρ_3	ρ_2	ρ_5	ρ_4	ρ_7	ρ_6
ρ_2 $(+,-,+,+,-,+)$	ρ_2	ρ_3	ρ_0	ρ_1	ρ_6	ρ_7	ρ_4	ρ_5
ρ_3 $(+,-,-,-,-,+)$	ρ_3	ρ_2	ρ_1	ρ_0	ρ_7	ρ_6	ρ_5	ρ_4
ρ_4 $(-,+,+,+,+,-)$	ρ_4	ρ_5	ρ_6	ρ_7	ρ_0	ρ_1	ρ_2	ρ_3
ρ_5 $(-,+,-,-,+,-)$	ρ_5	ρ_4	ρ_7	ρ_6	ρ_1	ρ_0	ρ_3	ρ_2
ρ_6 $(-,-,+,+,-,-)$	ρ_6	ρ_7	ρ_4	ρ_5	ρ_2	ρ_3	ρ_0	ρ_1
ρ_7 $(-,-,-,-,-,-)$	ρ_7	ρ_6	ρ_5	ρ_4	ρ_3	ρ_2	ρ_1	ρ_0

4.4 Symmetric permutation and sign inversion group G_6 derived from E_6

From the 6-dimensional symmetric permutation group G and the sign inversion group G_ρ shown above, we obtain its direct product G_6. The order of G_6 is 384 (48×8). As in the 4-dimensional case, we can extract from G_6 a normal subgroup H_6 that can be extended to a rotation group.

The order of this normal subgroup H is 96, and all the elements of G_6 are classified into the 4 residue classes C_0 to C_3 using modulus H. Table 14 shows these residue classes C_0 to C_3. Note that the signs ($\pm$) and ($\mp$) are double-sign corresponds. Next, we extend this finite normal subgroup H to a symmetric subgroup of the special orthogonal group $\mathcal{SO}(6)$ of 6th order, which is a topological group in a continuous group. Every element of a normal subgroup H can be represented in the form of a product of symmetric rotation pairs $R_{1a}(\theta)$, $R_{1b}(\theta)$, $R_{2a}(\theta)$, $R_{2b}(\theta)$, $R_3(\theta)$, and $R_4(\theta)$ as follows.

$$R_{1a}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{42}$$

$$R_{1b}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 & 0 & 0 \\ 0 & \sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{43}$$

$$R_{2a}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{44}$$

$$R_{2b}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & \sin\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{45}$$

Table. 14 Elements of the symmetric permutation and sign inversion group G_6 and classification by the residue classes (double-sign corresponds)

C_0 (= H)	C_1	C_2	C_3
±1,±2, 3, 4,±5,±6	±1,±2,-3,-4,±5,±6	±1,±2, 4, 3,±5,±6	±1,±2,-4,-3,±5,±6
±1,∓2,-3,-4,∓5,±6	±1,∓2, 3, 4,∓5,±6	±1,∓2,-4,-3,∓5,±6	±1,∓2, 4, 3,∓5,±6
±1,∓3, 2, 5,∓4,±6	±1,∓3,-2,-5,∓4,±6	±1,∓3, 5, 2,∓4,±6	±1,∓3,-5,-2,∓4,±6
±1,±3,-2,-5,±4,±6	±1,±3, 2, 5,±4,±6	±1,±3,-5,-2,±4,±6	±1,±3, 5, 2,±4,±6
±1,∓4, 5, 2,∓3,±6	±1,∓4,-5,-2,∓3,±6	±1,∓4, 2, 5,∓3,±6	±1,∓4,-2,-5,∓3,±6
±1,±4,-5,-2,±3,±6	±1,±4, 5, 2,±3,±6	±1,±4,-2,-5,±3,±6	±1,±4, 2, 5,±3,±6
±1,±5, 4, 3,±2,±6	±1,±5,-4,-3,±2,±6	±1,±5, 3, 4,±2,±6	±1,±5,-3,-4,±2,±6
±1,∓5,-4,-3,∓2,±6	±1,∓5, 4, 3,∓2,±6	±1,∓5,-3,-4,∓2,±6	±1,∓5, 3, 4,∓2,±6
±2,∓1, 3, 4,∓6,±5	±2,∓1,-3,-4,∓6,±5	±2,∓1, 4, 3,∓6,±5	±2,∓1,-4,-3,∓6,±5
±2,±1,-3,-4,±6,±5	±2,±1, 3, 4,±6,±5	±2,±1,-4,-3,±6,±5	±2,±1, 4, 3,±6,±5
±2,±3, 1, 6,±4,±5	±2,±3,-1,-6,±4,±5	±2,±3, 6, 1,±4,±5	±2,±3,-6,-1,±4,±5
±2,∓3,-1,-6,∓4,±5	±2,∓3, 1, 6,∓4,±5	±2,∓3,-6,-1,∓4,±5	±2,∓3, 6, 1,∓4,±5
±2,±4, 6, 1,±3,±5	±2,±4,-6,-1,±3,±5	±2,±4, 1, 6,±3,±5	±2,±4,-1,-6,±3,±5
±2,∓4,-6,-1,∓3,±5	±2,∓4, 6, 1,∓3,±5	±2,∓4,-1,-6,∓3,±5	±2,∓4, 1, 6,∓3,±5
±2,∓6, 4, 3,∓1,±5	±2,∓6,-4,-3,∓1,±5	±2,∓6, 3, 4,∓1,±5	±2,∓6,-3,-4,∓1,±5
±2,±6,-4,-3,±1,±5	±2,±6, 4, 3,±1,±5	±2,±6,-3,-4,±1,±5	±2,±6, 3, 4,±1,±5
±3,±1, 2, 5,±6,±4	±3,±1,-2,-5,±6,±4	±3,±1, 5, 2,±6,±4	±3,±1,-5,-2,±6,±4
±3,∓1,-2,-5,∓6,±4	±3,∓1, 2, 5,∓6,±4	±3,∓1,-5,-2,∓6,±4	±3,∓1, 5, 2,∓6,±4
±3,∓2, 1, 6,∓5,±4	±3,∓2,-1,-6,∓5,±4	±3,∓2, 6, 1,∓5,±4	±3,∓2,-6,-1,∓5,±4
±3,±2,-1,-6,±5,±4	±3,±2, 1, 6,±5,±4	±3,±2,-6,-1,±5,±4	±3,±2, 6, 1,±5,±4
±3,∓5, 6, 1,∓2,±4	±3,∓5,-6,-1,∓2,±4	±3,∓5, 1, 6,∓2,±4	±3,∓5,-1,-6,∓2,±4
±3,±5,-6,-1,±2,±4	±3,±5, 6, 1,±2,±4	±3,±5,-1,-6,±2,±4	±3,±5, 1, 6,±2,±4
±3,±6, 5, 2,±1,±4	±3,±6,-5,-2,±1,±4	±3,±6, 2, 5,±1,±4	±3,±6,-2,-5,±1,±4
±3,∓6,-5,-2,∓1,±4	±3,∓6, 5, 2,∓1,±4	±3,∓6,-2,-5,∓1,±4	±3,∓6, 2, 5,∓1,±4
±4,±1, 5, 2,±6,±3	±4,±1,-5,-2,±6,±3	±4,±1, 2, 5,±6,±3	±4,±1,-2,-5,±6,±3
±4,∓1,-5,-2,∓6,±3	±4,∓1, 5, 2,∓6,±3	±4,∓1,-2,-5,∓6,±3	±4,∓1, 2, 5,∓6,±3
±4,∓2, 6, 1,∓5,±3	±4,∓2,-6,-1,∓5,±3	±4,∓2, 1, 6,∓5,±3	±4,∓2,-1,-6,∓5,±3
±4,±2,-6,-1,±5,±3	±4,±2, 6, 1,±5,±3	±4,±2,-1,-6,±5,±3	±4,±2, 1, 6,±5,±3
±4,∓5, 1, 6,∓2,±3	±4,∓5,-1,-6,∓2,±3	±4,∓5, 6, 1,∓2,±3	±4,∓5,-6,-1,∓2,±3
±4,±5,-1,-6,±2,±3	±4,±5, 1, 6,±2,±3	±4,±5,-6,-1,±2,±3	±4,±5, 6, 1,±2,±3
±4,±6, 2, 5,±1,±3	±4,±6,-2,-5,±1,±3	±4,±6, 5, 2,±1,±3	±4,±6,-5,-2,±1,±3
±4,∓6,-2,-5,∓1,±3	±4,∓6, 2, 5,∓1,±3	±4,∓6,-5,-2,∓1,±3	±4,∓6, 5, 2,∓1,±3
±5,∓1, 4, 3,∓6,±2	±5,∓1,-4,-3,∓6,±2	±5,∓1, 3, 4,∓6,±2	±5,∓1,-3,-4,∓6,±2
±5,±1,-4,-3,±6,±2	±5,±1, 4, 3,±6,±2	±5,±1,-3,-4,±6,±2	±5,±1, 3, 4,±6,±2
±5,±3, 6, 1,±4,±2	±5,±3,-6,-1,±4,±2	±5,±3, 1, 6,±4,±2	±5,±3,-1,-6,±4,±2
±5,∓3,-6,-1,∓4,±2	±5,∓3, 6, 1,∓4,±2	±5,∓3,-1,-6,∓4,±2	±5,∓3, 1, 6,∓4,±2
±5,±4, 1, 6,±3,±2	±5,±4,-1,-6,±3,±2	±5,±4, 6, 1,±3,±2	±5,±4,-6,-1,±3,±2
±5,∓4,-1,-6,∓3,±2	±5,∓4, 1, 6,∓3,±2	±5,∓4,-6,-1,∓3,±2	±5,∓4, 6, 1,∓3,±2
±5,∓6, 3, 4,∓1,±2	±5,∓6,-3,-4,∓1,±2	±5,∓6, 4, 3,∓1,±2	±5,∓6,-4,-3,∓1,±2
±5,±6,-3,-4,±1,±2	±5,±6, 3, 4,±1,±2	±5,±6,-4,-3,±1,±2	±5,±6, 4, 3,±1,±2
±6,±2, 4, 3,±5,±1	±6,±2,-4,-3,±5,±1	±6,±2, 3, 4,±5,±1	±6,±2,-3,-4,±5,±1
±6,∓2,-4,-3,∓5,±1	±6,∓2, 4, 3,∓5,±1	±6,∓2,-3,-4,∓5,±1	±6,∓2, 3, 4,∓5,±1
±6,∓3, 5, 2,∓4,±1	±6,∓3,-5,-2,∓4,±1	±6,∓3, 2, 5,∓4,±1	±6,∓3,-2,-5,∓4,±1
±6,±3,-5,-2,±4,±1	±6,±3, 5, 2,±4,±1	±6,±3,-2,-5,±4,±1	±6,±3, 2, 5,±4,±1
±6,∓4, 2, 5,∓3,±1	±6,∓4,-2,-5,∓3,±1	±6,∓4, 5, 2,∓3,±1	±6,∓4,-5,-2,∓3,±1
±6,±4,-2,-5,±3,±1	±6,±4, 2, 5,±3,±1	±6,±4,-5,-2,±3,±1	±6,±4, 5, 2,±3,±1
±6,±5, 3, 4,±2,±1	±6,±5,-3,-4,±2,±1	±6,±5, 4, 3,±2,±1	±6,±5,-4,-3,±2,±1
±6,∓5,-3,-4,∓2,±1	±6,∓5, 3, 4,∓2,±1	±6,∓5,-4,-3,∓2,±1	±6,∓5, 4, 3,∓2,±1

$$R_3(\theta)=\begin{pmatrix} \cos\theta & 0 & 0 & -\sin\theta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & 0 & 0 & \sin\theta \\ \sin\theta & 0 & 0 & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix} \tag{46}$$

$$R_4(\theta)=\begin{pmatrix} \cos\theta & 0 & 0 & 0 & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & 0 & 0 & \sin\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sin\theta & 0 & 0 & 0 & \cos\theta & 0 \\ 0 & -\sin\theta & 0 & 0 & 0 & \cos\theta \end{pmatrix} \tag{47}$$

4.5 Basic rotation model with symmetric rotation pairs

Using the symmetric rotation pair $R_{1a}(\theta_1)$, $R_{1b}(\theta_2)$, $R_{2a}(\theta_3)$, $R_{2b}(\theta_4)$, $R_3(\theta_5)$, and $R_4(\theta_6)$, we define 4 basic rotation models I∼Ⅳ. BY this model, we can represent the 4 residue classes C_0∼C_3 by setting the parameters θ_1∼θ_6 of the symmetric rotation pairs appropriately.

4.5.1 Basic Rotation Model I ($C_0 = H$)

The reference rotation model (I) is expressed using continuous parameters as follows.

$$E_6 R_{1a}(\theta_1) R_{1b}(\theta_2) R_{2a}(\theta_3) R_{2b}(\theta_4) R_3(\theta_5) R_4(\theta_6)$$

4.5.2 Basic Rotation Model Ⅱ (C_1)

There are various variations of the basic rotation model corresponding to the residue class C_1. For example, using the left residue class of $\rho_1 \in C_1$, it can be expressed as follows.

$$E_6 \rho_1 R_{1a}(\theta_1) R_{1b}(\theta_2) R_{2a}(\theta_3) R_{2b}(\theta_4) R_3(\theta_5) R_4(\theta_6)$$

4.5.3 Basic Rotation Model Ⅲ (C_2)

The basic rotation model corresponding to the residue class C_2, for example, using the left residue class of $\sigma_a \in C_2$, is as follows.

$$E_6 \sigma_a R_{1a}(\theta_1) R_{1b}(\theta_2) R_{2a}(\theta_3) R_{2b}(\theta_4) R_3(\theta_5) R_4(\theta_6)$$

4.5.4 Basic Rotation Model Ⅳ (C_3)

The basic rotation model corresponding to the residue class C_3 is expressed as follows, using the left residue class of $\sigma_a\ \rho_1 \in\ C_3$.

$$E_6 \sigma_a \rho_1 R_{1a}(\theta_1) R_{1b}(\theta_2) R_{2a}(\theta_3) R_{2b}(\theta_4) R_3(\theta_5) R_4(\theta_6)$$

The right residue class can also be used to represent the basic rotation model Ⅱ∼Ⅳ. where I and Ⅱ of the basic rotation model are symmetric subgroups of the special orthogonal group $\mathcal{SO}(6)$ and I∼Ⅳ of the basic rotation model corresponds to a symmetric subgroup of the orthogonal group $\mathcal{O}(6)$[15]. Note that unlike the 4-dimension case, the product of these symmetric rotation pairs is noncommutative, and the values of the rotation parameters are not preserved if the order of the products is exchanged. However, by choosing appropriate values, the order of products of symmetric rotation pairs can be exchanged without changing the shape of the basis[13].

5 Rotation model of 6-dimensional generalized LOT

This chapter describes how to construct a 6-dimensional generalized LOT rotation model.

5.1 Construction method for 6-dimensional LOT (6×12)

In the previous section, we showed that for the 6 columns of the basic symmetric matrix E_6, all orthonormal bases satisfying the 3 conditions ① symmetry, ② orthogonality, and ③ norm 1, can be generated by adding 4 operations corresponding to the right basic deformation.

Applying the same operation to the above basis with a shift of 3 columns, the length of the orthogonal basis is extended to 12 columns of 2 blocks. This operation generates all the bases of LOT(6×12) that satisfy the above 3 conditions[13]. For each of the 4 types of operations I∼Ⅳ in the 1st stage shown in the previous section, the rotation operations in the 2nd stage I∼Ⅳ are added, so the rotation model is extended to 16 types of (I−I)∼(Ⅳ−Ⅳ). Figure 8 shows an example of the model (Ⅳ−Ⅳ). In this example, the right residue class representation is used, but it can also be expressed in terms of the left residue class.

5.2 Equivalent conversion rule for coding gains in LOT(6×12)

As in the 4-dimensional case, the ρ_1 operation can be moved upstage, so the following equivalent transformation rules for the coding gain are satisfied.

$$\mathrm{I} \;\Leftrightarrow\; \mathrm{II} \tag{48}$$

$$\mathrm{III} \;\Leftrightarrow\; \mathrm{IV} \tag{49}$$

5.3 Construction method for 6-dimensional GenLOT (6×18)

By adding operations such as the rotation of the 3rd stage (I∼Ⅳ) to the rotational model of LOT (6×12), the base length is extended from 12 to 18. That is, bases of the generalized LOT(6×18) can be constructed.

Figure 9 shows an example of the rotation model (Ⅲ−I−Ⅲ). Note that R_{1a-3},R_{1b-3},R_{2a-3} in the 3rd stage can be

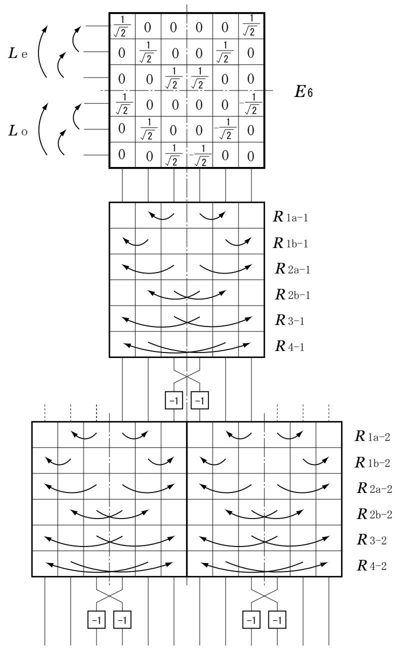


Fig. 8 Rotation model of LOT(6×12)(Ⅳ−Ⅳ)

finally absorbed into R_{1a-2},R_{1b-2},R_{2a-2} by moving to the 2nd stage. Furthermore, R_{1a-2},R_{1b-2}, and R_{2a-2} in the 2nd stage is equivalent to R_{1b-1},R_{2b-1}, and R_{3-1} in the 1st stage, respectively, and can be finally integrated into the left basic deformation. In this case the minimum number of rotation parameters is 12: 3 for R_{2b-3},R_{3-3}, and R_{4-3} in the 3rd stage, 3 for R_{2b-2},R_{3-2}, and R_{4-2} in the 2nd stage, 3 for the left basic deformation L_e, and 3 for L_o[13].

5.4 Equivalent conversion rule for coding gains in generalized LOT (6×18)

For the rotation model (Ⅲ−I−Ⅲ) in Figure 9, by moving the cross section (σ_a) of the 1st and 3rd stages to the 2nd stage, an equivalent transformation is made as shown in Figure 10. At this time, there remain operations above and below the 2nd stage that correspond to permutations shifted by one block (6 rows). That is, it is not equivalent to (I−I−I) of the model as in the 4-dimensional case. When the right basic deformation of the rotation model (Ⅲ−I−Ⅲ) is given the values 0,$\pm\frac{\pi}{2}$,π as the rotation angles of the symmetric rotation pairs, 1248 patterns appear in total. These patterns were confirmed by simulation to be in perfect agreement with the patterns in the model (I−I−I). This shows that an equivalent transformation for the coding gain is established between the model (I−I−I) and the model (Ⅲ−I−Ⅲ). Furthermore, by adding a transposition σ_a to the lower part of the 3rd stage, it becomes clear that

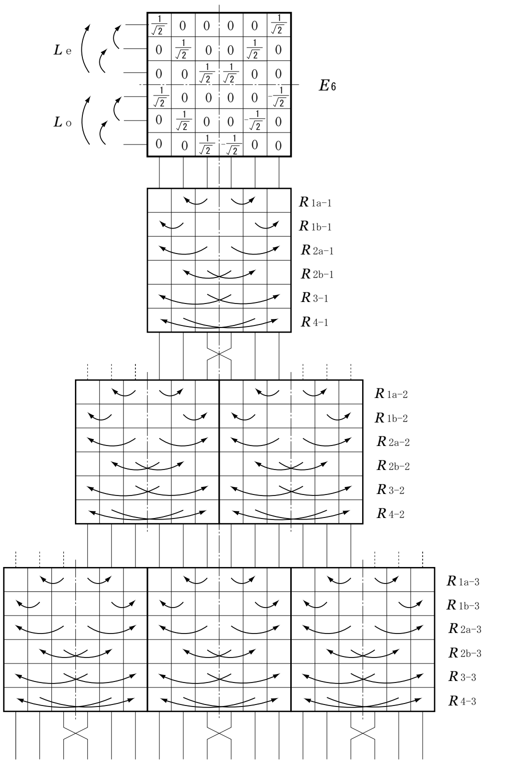


Fig. 9 Rotation model of GenLOT(6 × 18)(Ⅲ−I−Ⅲ)

model (I−I−Ⅲ) and model (Ⅲ−I−I) are equivalent. From this, the equivalent conversion rule for the coding gain in 6 dimensions is as follows.

$$\mathrm{I}-\mathrm{I}-\mathrm{I} \;\Leftrightarrow\; \mathrm{III}-\mathrm{I}-\mathrm{III} \tag{50}$$

$$\mathrm{I}-\mathrm{I}-\mathrm{III} \;\Leftrightarrow\; \mathrm{III}-\mathrm{I}-\mathrm{I} \tag{51}$$

Next, the parameters that maximize the coding gain for the generalized LOT of (6×12)∼(6×24) were obtained by simulation. The results are shown in Table 15. Note that the 1st stage is optional, so the basic (I) is chosen.

For example, if we optimize the parameters for all models in (6×12) and find the maximum value of the coding gain, we can divide them into 2 groups. Similarly, the (6×18) case is classified into 3 groups, and the (6×24) case into 4 groups, which are confirmed to follow the equivalent transformation rules in (50),(51).

6 Rotation models of generalized LOT over 8 dimensions

6.1 Rotation models of generalized LOT

The 8th-order symmetry equation is shown below.

$$f_8 = x_1\ x_8 + x_2\ x_7 + x_3\ x_6 + x_4\ x_5 \tag{52}$$

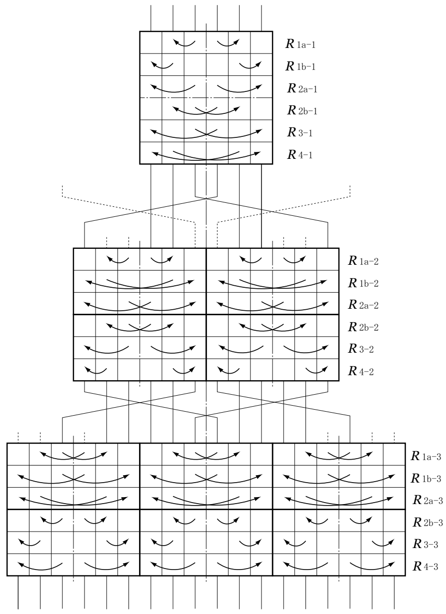


Fig. 10 Equivalent transformation rules on 6 dimensional GenLOT (Ⅲ−I−Ⅲ) ⇔ (I−I−I)

Table. 15 Optimum parameters of 6 dimensional GenLOT(6 × 12)∼(6 × 24) and classifications

Size of basis	Stage 2nd	Stage 3rd	Stage 4th	Coding gain max value(dB)
(6 × 12)	I (II)			8.854
	III (IV)			8.825
(6 × 18)	I (II)	I (II)		9.019
		III (IV)		
	III (IV)	I (II)		8.888
		III (IV)		9.005
(6 × 24)	I (II)	I (II)	I (II)	9.123
		III (IV)	I (II)	
	III (IV)	I (II)	III (IV)	
	I (II)	I (II)	III (IV)	9.107
		III (IV)	III (IV)	
	III (IV)	I (II)	I (II)	
		III (IV)	I (II)	9.062
			III (IV)	9.035

In this case, as shown in Table 16, the order of the symmetric permutation group G_σ is 384, and that of the sign inversion group G_ρ is 16. The order of the symmetric permutation and sign inversion group G represented by the direct product is 6144. The order of the normal subgroup H of G, which can be extended to a rotation group, is 1536. Also, the number of residue class formed by modulus H is 4.

In the same way, the orders of G and H in the $2n$-dimensional generalized LOT are obtained. However, in both cases, the number of residue classes is 4, and the number of models per stage is also 4.

Table. 16 Orders of the groups on GenLOT and its classifications

	Dimension of GenLOT			
	4	6	8	$2n$
Symmetric permutation group G_σ	8	48	384	$n!2^n$
Sign inversion group G_ρ	4	8	16	2^n
Symmetric permutation・sign inversion group G	32	384	6144	$n!2^{2n}$
Normal subgroup of G H	8	96	1536	$n!2^{2(n-1)}$
Number of column rotation parameters	2	6	12	$n(n-1)$
residue class of G for modulus H	4			

6.2 Equivalent transformation rules for coding gain

In the rotation model of the generalized LOT, when the angles of the symmetric rotation pairs are set to 0, $\pm\frac{\pi}{2}$,π as in Tables 7 and 8, the number of patterns for the sequence of columns of the basic symmetric matrix E is organized as shown in Table 17. Note that, of course, the 1st stage is equal to the order of H. In the 8-dimensional 3rd stage, a combination of rotation parameters transforms the 8-column sequence of the basic symmetric matrix E_8 into 61440 different patterns. For the models (I−Ⅲ−I) and (Ⅲ−Ⅲ−Ⅲ), we compared their patterns by computer simulation and found them to be in perfect agreement. This shows that the maximum values of these coding gains are equal and that the same equivalent transformation rules (32)〜(35) hold as for the 4-dimension GenLOT.

Table. 17 The numbers of rotation model patterns with fixed value of rotation angles

Rotation model	Dimension of GenLOT			
	4	6	8	$2n$
1st stage	8	96	1536	$n!2^{2(n-1)}$
2nd stage	16	384	12288	$n!2^{3(n-1)}$
3rd stage	32	1248	61440	$n!2^{2(n-1)}\sum_{i=1}^{n} 3^{i-1}$

Equivalent conversion rules for coding gain are shown in Table 18. Comparing the (8)-dimensional and 6-dimensional rules, (I) and (Ⅲ) are interchanged.

The generalized LOT in 10 dimensions has a large number of combinations, and its validation is an issue to be addressed in the future. However, it is expected to be consistent with 6 dimensions.

Table. 18 Equivalent transformation rules on coding gain of GenLOT

Dimension	Equivalent transformation rules					
4, 8 $[4n]$	I	⇔	Ⅱ	I−Ⅲ−I	⇔	Ⅲ−Ⅲ−Ⅲ
				I−Ⅲ−Ⅲ	⇔	Ⅲ−Ⅲ−I
6 $[2(2n+1)]$	Ⅲ	⇔	Ⅳ	Ⅲ−I−Ⅲ	⇔	I−I−I
				Ⅲ−I−I	⇔	I−I−Ⅲ

7 Conclusion

We have identified methods for constructing rotational models of linear phase generalized LOTs in which all orthonormal bases can be represented. In order to concisely describe all combinations of operations such as rotation and permutation of the LOT basis, we define a finite symmetric permutation group and a sign inversion group for the columns of a basic symmetric matrix, and extract a normal subgroup H from their direct product G that can be extended to a continuous rotation group. Next, we proposed methods to integrate redundant operations existing between stages by classifying the elements of G into 4 residue classes using modulus H and generating rotation models corresponding to them.

Since the variation of this rotation model increases by a factor of 4 per stage, a method to efficiently search for the optimal parameters is required. We focused on the property that the coding gain, which is widely used as a measure of coding efficiency, is invariant to operations such as LOT basis substitution, sign reversal of ±, and mirroring, and clarified the equivalent transformation rules between stages whose optimal values are preserved in 4-dimensional and 6-dimensional rotation models. Furthermore, by organizing and integrating the above rotation model using the 4 extracted rules, it was shown that the model can be classified into groups equal to the number of stages

This allows the design of generalized LOTs to be optimized using the fewest number of parameters for a representative model equal to the number of stages, greatly reducing the amount of work required for searching, etc. Future work is to verify the equivalent transformation rule of the coding gain for generalized LOTs of 10 or more dimensions, and to extend this method to odd dimensions

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