Infinitesimal locally trivial deformation spaces of compact complex surfaces with ordinary singularities

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Abstract: Let S be a compact complex surface with *ordinary singularities*. We denote by Θ_S the sheaf of germs of holomorphic tangent vector fields on S. In this paper we shall give a description of the cohomology $H^1(S, \Theta_S)$, which is called the *infinitesimal locally trivial deformation space* of S, using a 2-cubic hyper-resolution of S in the sense of F. Guillén, V. Navarro Aznar *et al.* ([1]). As a by-product, we shall show that the natural homomorphism $H^1(S, \Theta_S) \to H^1(X, \Theta_X(-\log D_X))$ is injective under some condition, where X is the (non-singular) normal model of S, D_X the inverse image of the double curve D_S of S by the normalization map $f : X \to S$, and $\Theta_X(-\log D_X)$ the sheaf of germs of logarithmic tangent vector fields along D_X on X. Note that the cohomology $H^1(X, \Theta_X(-\log D_X))$ is nothing but the infinitesimal locally trivial deformation space of a pair (X, D_X).

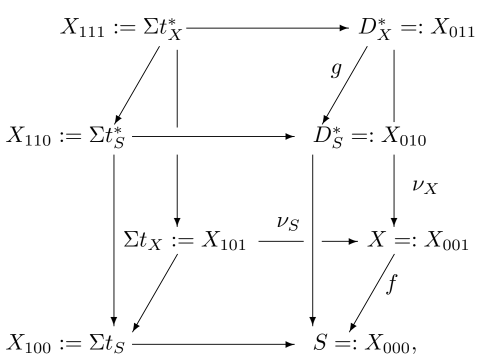
§1. 2-cubic hyper-resolutions of compact complex surfaces with ordinary singularities. A 2-dimensional compact complex space S is called a compact complex surface with *ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurfaces at the origin of the complex 3-space $\mathbf{C}^3$ at every point of S:

(i) $z = 0$ (simple point),
(ii) $yz = 0$ (ordinary double point),
(iii) $xyz = 0$ (ordinary triple point),
(iv) $xy^2 - z^2 = 0$ (cuspidal point),

where (x, y, z) is the coordinate on $\mathbf{C}^3$. These surfaces are attractive because every smooth complex projective surface can be obtained as the normalization of such a surface S in the 3-dimensional complex projective space $P^3(\mathbf{C})$. In fact, every smooth, compact complex surface embedded in a complex projective space can be projected onto such a surface S in $P^3(\mathbf{C})$ via generic projection. We denote by D_S the singular locus of S, and call it the *double curve* of S. D_S is a singular curve with triple points. We denote by Σt_S the triple point locus of S, and by $\Sigma \mathfrak{c}_S$ the cuspidal point locus of S. Let $f : X \to S$ be the normalization. Note that X is non-singular. We put $D_X := f^{-1}(D_S)$ and $\Sigma t_X := f^{-1}(\Sigma t_S)$. D_X is a singular curve with nodes and Σt_X coincides with the set of nodes of D_X. Let $n_S : D_S^* \to D_S$ and $n_X : D_X^* \to D_X$ be the normalizations, and let $g : D_X^* \to D_S^*$ be the lifting of the map $f_{|D_X} : D_X \to D_S$. We put $\Sigma t_S^* := n_S^{-1}(\Sigma t_S)$ and $\Sigma t_X^* := n_X^{-1}(\Sigma t_X)$. Then a 2-*cubic hyper-resolution* of S in the sense of F. Guillén, V. Navarro Aznar *et al.* ([1]) is obtained as in the diagram $(*)$ below. In the diagram, ν_S and ν_X are the composites of the normalizations and the inclusion maps, and the square on the left-hand side is the one induced from the square on the right-hand side.

§2. Description of $H^1(S, \Theta_S)$ by use of a 2-cubic hyper-resolution of S. We put $\Theta_S := \mathcal{H}om_{\mathcal{O}_S}(\Omega_S^1, \mathcal{O}_S)$, and call it the *sheaf of germs of holomorphic tangent vector fields on S*. We call $H^1(S, \Theta_S)$ the *infinitesimal locally trivial deformation space* of a compact complex surface S with ordinary singularities. This naming is due to the fact that the parameter space of the 1st-order infinitesimal *locally trivial deformation* of S sits in this space, where "*locally trivial deformation*" means the deformation which preserves local analytic singularity types. In the following we shall describe $H^1(S, \Theta_S)$ by use of the diagram $(*)$. We denote symbolically the 2-cubic hyper-resolution of S in the diagram $(*)$ by $b.: X. \to S$. For each $\alpha \in \mathrm{Ob}(\square_2^+) := \{\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbf{Z}^3 \mid 0 \le \alpha_i \le 1 \text{ for } 0 \le i \le 2\}$, an object of the *augmented 2-cubic category* in the sense of F. Guillén, V. Navarro Aznar et al. ([1]), we denote

(*)



by Θ_{X_α} the sheaf of germs of holomorphic tangent vector fields on X_α ($X_0 := S$ for $0 := (0,0,0) \in \mathrm{Ob}\square_2^+$), and by $\Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ the sheaf of germs of $\mathcal{O}_{X_\alpha}$-valued derivations on S, i.e., $\theta \in \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ is a $\mathbf{C}$-linear map $\mathcal{O}_S \to b_{\alpha *}\mathcal{O}_{X_\alpha}$ with the property $\theta(uv) = \theta(u)v + u\theta(v)$ for $u, v \in \mathcal{O}_S$, where b_α is the map from X_α to S in the diagram (*) (cf. [2]). For each $\alpha \in \mathrm{Ob}(\square_2) := \{\alpha \in \mathrm{Ob}(\square_2^+) \mid \alpha \neq (0,0,0)\}$, we define $tb_\alpha : b_{\alpha *}\Theta_{X_\alpha} \to \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ (resp. $\omega b_\alpha : \Theta_S \to \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$) by $tb_\alpha(\theta) := \theta b_\alpha^*$ *for* $\theta \in b_{\alpha *}\Theta_{X_\alpha}$ (resp. $\omega b_\alpha(\varphi) := b_\alpha^* \varphi$ *for* $\varphi \in \Theta_S$), where $b_\alpha^* : \mathcal{O}_S \to b_{\alpha *}\mathcal{O}_{X_\alpha}$ denotes the pull-back.

Definition 1. We define a sheaf $\Theta(b.)$ to be

$$\mathrm{Ker}\{\oplus_{\alpha \in \mathrm{Ob}(\square_2^+)} b_{\alpha *}\Theta_{X_\alpha} \to \oplus_{\alpha \in \mathrm{Ob}(\square_2)} \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha}) : (\theta_\alpha) \to tb_\alpha(\theta_\alpha) - \omega b_\alpha(\theta_0)\},$$

and call it the *sheaf of germs of holomorphic tangent vector fields to the* 2*-cubic hyper-resolution* $b. : X. \to S$.

Further, we introduce the following notation:

$\Theta_X(-\log D_X)$: the sheaf of germs of logarithmic tangent vector fields along D_X on X, i.e., the subsheaf of Θ_X consisting of derivations of $\mathcal{O}_X$ which send $\mathcal{I}(D_X)$, the ideal sheaf of D_X in $\mathcal{O}_X$, into itself,

$\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)$: the sheaf of germs of holomorphic tangent vector fields on D_S^* which vanish on $\Sigma\mathfrak{c}_S^*$ and Σt_S^*, where $\Sigma\mathfrak{c}_S^*$ is the inverse image of the cuspidal point locus $\Sigma\mathfrak{c}_S$ of S by the normalization map $n_S : D_S^* \to D_S$,

$\Theta_{D_X^*}(-\Sigma t_X^*)$: the sheaf of germs of holomorphic tangent vector fields on D_X^* which vanish on Σt_X^*. (Note that Σt_X^* coincides with the inverse image of the triple point locus Σt_S of D_S by the composed map $n_S \circ g : D_X^* \to D_S$.)

Proposition 2. *There exists naturally the following exact sequence of* $\mathcal{O}_S$*-modules*:

$$0 \to \Theta_S \overset{\widehat{\omega f} \oplus \widehat{\omega \nu_S}}{\longrightarrow} f_*\Theta_X(-\log D_X) \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*) \overset{\widehat{\omega\nu_X} - \widehat{\omega g}}{\longrightarrow} \nu_*\Theta_{D_X^*}(-\Sigma t_X^*) \to 0,$$

where $\nu := f \circ \nu_X = \nu_S \circ g$.

The proof of this proposition is a direct calculation by use of the local coordinate description of the maps $f : X \to S$, $\nu_S : D_S^* \to D_S$, $\nu_X : D_X^* \to X$, and $g : D_X^* \to D_S^*$.

Corollary 3. $\Theta(b.) \simeq \Theta_S$.

Theorem 4. *If the map*

$$H^0(X, \Theta_X(-\log D_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)) \to H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is surjective, then we have

$$H^1(S, \Theta(b.)) \simeq H^1(S, \Theta_S) \simeq \textit{the kernel of the map}$$
$$H^1(X, \Theta_X(-\log D_X)) \oplus H^1(D_S^*, \Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)) \to H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*)).$$

Proposition 5. *The map*

$$H^1(D_S^*, \Theta_{D_S^*}(-\Sigma\mathfrak{c}_S^* - \Sigma t_S^*)) \to H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is injective.

The proof of this proposition will be completed after a few lemmas. First, we will prove general facts about a double covering $\pi : C_1 \to C$ between compact Riemann surfaces, or connected, compact complex manifolds of dimension 1. We denote by $\Sigma\mathfrak{c}$ the branch locus of the double covering $\pi : C_1 \to C$, and by $[\Sigma\mathfrak{c}]$ the line bundle over C determined by the divisor $\Sigma\mathfrak{c}$. Due to Wavrik's result ([6]), there exists a complex line bundle F over C such that;

(i) $F^{\otimes 2} = [\Sigma\mathfrak{c}]$, and
(ii) C_1 is a submanifold of F and the bundle map $F \to C$ realizes the double covering $\pi : C_1 \to C$.

Lemma 6. *With the notation above, there exists an exact sequence of* $\mathcal{O}_C$*-modules*

$$(2.1) \quad 0 \to \mathcal{O}_C \to \pi_*\mathcal{O}_{C_1} \to \mathcal{O}_C(F^{-1}) \to 0.$$

This follows from the concrete description of the transition functions of the line bundle F by use of local coordinates.

Let $\pi : C_1 \to C$ and $\Sigma\mathfrak{c}$ be the same as before, and let Σt be a set of finite distinct points of C with $\Sigma\mathfrak{c} \cap \Sigma t = \emptyset$. We put $\Sigma t_1 := \pi^{-1}(\Sigma t)$.

Lemma 7. *With the notation above, we have* *an exact sequence of $\mathcal{O}_C$-modules*

$$(2.2)\quad 0 \to \Theta_C(-\Sigma\mathfrak{c} - \Sigma t) \to \pi_*\Theta_{C_1}(-\Sigma t_1) \to \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}) \to 0.$$

Proof . Since $\pi_*(\pi^*\Theta_C(-\Sigma t)) \simeq \Theta_C(-\Sigma t) \otimes \pi_*\mathcal{O}_{C_1}$, tensoring the sheaf $\Theta_C(-\Sigma t)$ to the exact sequence in (2.1), we have an exact sequence of $\mathcal{O}_C$-modules

$$(2.3)\quad 0 \to \Theta_C(-\Sigma t) \to \pi_*(\pi^*\Theta_C(-\Sigma t)) \to \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}) \to 0.$$

We also have the following commutative diagram of exact sequences of $\mathcal{O}_C$-modules:

$$\begin{array}{ccccc} 0 & \longrightarrow & \Theta_C(-\Sigma t) & \xrightarrow{\widehat{\omega\pi}} & \pi_*(\pi^*\Theta_C(-\Sigma t)) \\ & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \Theta_C(-\Sigma\mathfrak{c} - \Sigma t) & \longrightarrow & \pi_*\Theta_{C_1}(-\Sigma t_1) \\ & & \uparrow & & \uparrow \\ & & 0 & & 0, \end{array}$$

where $\widehat{\omega\pi}$ denotes the pull-back. We will show that this diagram gives an isomorphism

$$(2.4)\quad \pi_*\Theta_{C_1}(-\Sigma t_1)/\Theta_C(-\Sigma\mathfrak{c} - \Sigma t) \simeq \pi_*(\pi^*\Theta_C(-\Sigma t))/\Theta_C(-\Sigma t).$$

To prove the surjectivity of the homomorphism in (2.4), we will first show that

$$(2.5)\quad \widehat{t\pi}(\Theta_{C_1}(-\Sigma t_1)_{\pi^{-1}(p)}) + \widehat{\omega\pi}(\Theta_C(-\Sigma t)_p) = \pi^*\Theta_C(-\Sigma t)_{\pi^{-1}(p)}$$

for any point $p \in C$, where $\widehat{t\pi}$ denotes the map derived from the Jacobian map of the map π. If $p \notin \Sigma\mathfrak{c}$, (2.5) obviously holds. Assume $p \in \Sigma\mathfrak{c}$. We put $q := \pi^{-1}(p)$, and let u and v be local coordinates around p and q with center p and q, respectively. We may assume that the map $\pi : C_1 \to C$ is given by $v \to u = v^2$ at q. For a local cross-section $a(v)\pi^*(\partial/\partial u)$ of $\pi^*\Theta_C(-\Sigma t)$ at q where $a(v)$ is a holomorphic function of v, we express $a(v)$ as

$$a(v) = a(0) + va_1(v)$$

where $a_1(v)$ is a holomorphic function of v. Then we have

$$\widehat{t\pi}\left(\frac{1}{2}a_1(v)\left(\frac{\partial}{\partial v}\right)\right) + \widehat{\omega\pi}\left(a(0)\left(\frac{\partial}{\partial u}\right)\right) = (va_1(v) + a(0))\pi^*\left(\frac{\partial}{\partial u}\right) = a(v)\pi^*\left(\frac{\partial}{\partial u}\right),$$

which shows (2.5) holds for the point $p \in \Sigma\mathfrak{c}$. To prove the injectivity of the homomorphism in (2.4), it suffices to show that, for any point $p \in C$ and a local holomorphic cross-section θ_1 of $\pi_*\Theta_{C_1}(-\Sigma t_1)$ at p, if $\widehat{t\pi}(\theta_{1,p})$ belongs to $\widehat{\omega\pi}(\Theta_C(-\Sigma t)_p)$, then $\theta_{1,p}$ belongs to the image $\Theta_C(-\Sigma\mathfrak{c} - \Sigma t)_p$ in $\pi_*\Theta_{C_1}(-\Sigma t_1)_p$. Since this is obvious if $p \notin \Sigma\mathfrak{c}$, we assume $p \in \Sigma\mathfrak{c}$. We take the same local coordinates u and v around p and $q := \pi^{-1}(p)$ as before, respectively. For a local cross-section $\theta_1 = a_1(v)(\partial/\partial v)$ of $\Theta_{C_1}(-\Sigma t_1)$ at q, we assume that there exists a local cross-section $\theta = a(u)(\partial/\partial u)$ of $\Theta_C(-\Sigma t)$ at p such that $\hat{t\pi}(\theta_1) = \hat{\omega\pi}(\theta)$. Then

$$2a_1(v)v\pi^*\left(\frac{\partial}{\partial v}\right) = a(v^2)\pi^*\left(\frac{\partial}{\partial v}\right)$$

Hence $a(0) = 0$, that is, θ belongs to $\Theta_C(-\Sigma t - \Sigma\mathfrak{c})$. This means θ_1 belongs to the image of $\Theta_C(-\Sigma\mathfrak{c} - \Sigma t)$ in $\pi_*\Theta_{C_1}(-\Sigma t_1)$ at p. Now the exact sequence in (2.2) follows from (2.3) and (2.4). □

Remark 8. In the proof of Lemma 7, the equality in (2.5) is essential. This equality tells that the double branched covering map $\pi : C_1 \to C$ is locally stable in the sense of J. N. Mather.

Proof of Proposition 5. We may assume that D_S^* is irreducible, and so it suffices to show that the homomorphism

$$(2.6)\quad H^1(C, \Theta_C(-\Sigma\mathfrak{c} - \Sigma t)) \to H^1(C_1, \Theta_{C_1}(-\Sigma t_1))$$

derived from the exact sequence in (2.2) is injective. For this purpose, we count the degree of the line bundle $\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})$. We denote by $\mathfrak{K}_C$ and $g(C)$ the canonical line bundle and the genus of the curve C, respectively. Then, since $F^{\otimes 2} = \mathcal{O}_C([\Sigma\mathfrak{c}])$, we have

$$\begin{aligned} &\deg(\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) \\ &\quad = -\deg\mathfrak{K}_C - \deg F - \#\Sigma t \\ &\quad = -2(g(C) - 1) - \frac{1}{2}\#\Sigma\mathfrak{c} - \#\Sigma t, \end{aligned}$$

where # denote the cardinal numbers of sets. Then we have

$$-2(g(C) - 1) - \frac{1}{2}\#\Sigma\mathfrak{c} - \#\Sigma t < 0$$

with the exception of the following cases:

(i) $g(C) = 1,\ \Sigma\mathfrak{c} = \emptyset,\ \text{and}\ \Sigma t = \emptyset,$

(ii) $g(C) = 0,\ \Sigma\mathfrak{c} = \emptyset,\ \text{and}\ 0 \le \#\Sigma t \le 2,$

(iii) $g(C) = 0,\ \#\Sigma\mathfrak{c} = 2,\ \text{and}\ 0 \le \#\Sigma t \le 1,$

(iv) $g(C) = 0,\ \#\Sigma\mathfrak{c} = 4,\ \text{and}\ \Sigma t = \emptyset.$

Hence, excluding the exceptional cases listed above, we have

$$(2.7)\quad H^0(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) = 0,$$

and so the homomorphism in (2.6) is injective as required. Now, checking the exceptional cases, case by case, we conclude that the homomorphism in (2.6) is always injective. □

Corollary 9. *If the map*

$$H^0(X, \Theta_X(-\log D_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma \mathfrak{c}_S^* - \Sigma t_S^*)) \to H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is surjective, then the natural map

$$H^1(S, \Theta_S) \to H^1(X, \Theta_X(-\log D_X))$$

is injective.

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