42. Deformations of Complex Analytic Subspaces with Locally Stable Parametrizations of Compact Complex Manifolds

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Introduction. In this paper we shall give a definition of *complex analytic subspaces with locally stable parametrizations* of compact complex manifolds, which is a generalization of closed complex analytic subsets of *simple normal crossing* in [3] and analytic subvarieties with *ordinary singularities* in [8], and we show that their *logarithmic deformations* and *locally trivial displacements* are equivalent to deformations of *locally stable holomorphic maps* (cf. Definition 1.1 below). From this equivalence and Miyajima-Namba-Flenner's theorem on the existence of the Kuranishi family of deformations of holomorphic maps, it follows that there exist the Kuranishi family of logarithmic deformations and the maximal family of locally trivial displacements of a complex analytic subspace with a locally stable parametrization. These are a unification and a generalization of the results in [3] and [8]. Throughout this paper all complex analytic spaces are assumed to be reduced, second countable, and finite dimensional. For notation and terminology concerning logarithmic deformations, locally trivial displacements of a complex analytic subspace and deformations of a holomorphic map, we refer to [3], [8] and [2], respectively.

§1. Complex analytic subspaces with locally stable parametrizations and their deformations. Let X and Y be complex manifolds, and S and T finite subsets of X and Y, respectively. A multi-germ $f:(X,S)\to(Y,T)$ of a holomorphic map at S is an equivalence class of holomorphic maps $g:U\to Y$ with $g(S)=T$, where U are open neighborhoods of S in X. Throughout this paper we shall interchangeably use a multi-germ of f and a representative g of f. A germ of a parametrized family of multi-germs of holomorphic maps is a multi-germ $F:(X\times\mathbb{C}^r, S\times 0)\to(Y\times\mathbb{C}^r, T\times 0)$ of a holomorphic map such that $F(X\times t)\subset Y\times t$ for any t in some open neighborhood of 0 in $\mathbb{C}^r$. An unfolding of a multi-germ $f:(X,S)\to(Y,T)$ of a holomorphic map is a germ of a parametrized family of multi-germs of holomorphic maps $F:(X\times\mathbb{C}^r, S\times 0)\to(Y\times\mathbb{C}^r, T\times 0)$ such that $F(x,0)=(f(x),0)$ for $x\in X$. We say that an unfolding $F:(X\times\mathbb{C}^r, S\times 0)\to(Y\times\mathbb{C}^r, T\times 0)$ of a multi-germ $f:(X,S)\to(Y,T)$ of a holomorphic map is *trivial* if there exist germs of t-levels ($t\in\mathbb{C}^r$) preserving analytic automorphisms $G:(X\times\mathbb{C}^r, S\times 0)\to(X\times\mathbb{C}^r, S\times 0)$ and $H:(Y\times\mathbb{C}^r, T\times 0)\to(Y\times\mathbb{C}^r, T\times 0)$ with $G_{|X\times 0}=id_X$, $H_{|Y\times 0}=id_Y$, such that $H\circ F\circ G^{-1}=f\times id_{\mathbb{C}^r}$. We say that a multi-germ

$f:(X,S)\to(Y,T)$ of a holomorphic map is *simultaneously stable* if any unfolding of f is trivial.

1.1. Definition. A holomorphic map $f: X\to Y$ between complex manifolds is said to be *locally stable* if, for any point $y\in Y$ and any finite subset $S\subset f^{-1}(y)$, a multi-germ $f:(X,S)\to(Y,y)$ is simultaneously stable.

1.2. Definition. A complex analytic subspace Z of a complex manifold Y is said to be *with a locally stable parametrization* if

(i) the normal model X of Z is non-singular, and

(ii) the composite map $f:=\iota\circ n: X\to Y$ is locally stable, where $n: X\to Z$ is the normalization map and $\iota: Z\subset Y$ is the inclusion map.

From now on let Z be a complex analytic subspace with a locally stable parametrization of a compact complex manifold Y. For a pair (Y,Z), let $f:=\iota\circ n: X\to Y$ be the same as in Definition 1.2. We denote by $\mathcal{D}(f,X,Y)$ (resp. $\mathcal{D}(f,X)$) the category of germs of families of deformations of $f: X\to Y$ with Y varied (resp. with Y fixed), and by $\mathcal{L}(Y,Z)$ (resp. $\mathcal{L}(Z)$) the category of germs of families of logarithmic deformations of (Y,Z) (resp. of locally trivial displacements of Z in Y).

1.3. Theorem. *$\mathcal{D}(f,X,Y)$ and $\mathcal{L}(Y,Z)$ (resp. $\mathcal{D}(f,X)$ and $\mathcal{L}(Z)$) are isomorphic as categories.*

Proof. The proof is almost identical with that of Theorem (11.1) in [8] (=Main theorem in [7]). Although in [8] we consider only locally trivial displacements of Z in a fixed ambient manifold Y and deformations of $f: X\to Y$ with Y fixed, the proof of Theorem (11.1) in [8] is also valid for logarithmic deformations of a pair (Y,Z) and for deformations of $f: X\to Y$ with Y varied. Q.E.D.

§ 2. Comparison of infinitesimal deformation spaces. As in the preceding section, let Z be an analytic subspace with a locally stable parametrization in a compact complex manifold Y, and let $f:=\iota\circ n: X\to Y$ be the composite of the normalization map $n: X\to Z$ and the inclusion map $\iota: Z\subset Y$. We denote by T_Y the sheaf of holomorphic tangent vector fields on Y, and by $T_Y(\log Z)$ the sheaf of *logarithmic tangent vector fields along Z in Y*, that is, the subsheaf of T_Y consisting of the derivations of $\mathcal{O}_Y$ which send the ideal sheaf of Z in $\mathcal{O}_Y$ into itself. We define a sheaf $\mathcal{N}_{Z/Y}$ by the following exact sequence:

$$0\longrightarrow T_Y(\log Z)\longrightarrow T_Y\longrightarrow \mathcal{N}_{Z/Y}\longrightarrow 0, \tag{2.1}$$

and $\mathcal{T}_{X/Y}$ by the following one:

$$0\longrightarrow T_X\xrightarrow{Jf} f^*T_Y\longrightarrow \mathcal{T}_{X/Y}\longrightarrow 0. \tag{2.2}$$

The infinitesimal deformation spaces of logarithmic deformations of a pair (Y,Z), locally trivial displacements of Z in Y and deformations of a map $f: X\to Y$ with Y fixed, are $H^1(Y,T_Y(\log Z))$, $H^0(Z,\mathcal{N}_{Z/Y})$ and $H^0(X,\mathcal{T}_{X/Y})$, respectively. Their obstruction classes belong to $H^2(Y,T_Y(\log Z))$, $H^1(Z,\mathcal{N}_{Z/Y})$ and $H^1(X,\mathcal{T}_{X/Y})$, respectively. As to the infinitesimal deformation space of a map $f: X\to Y$ with Y varied, there are two spaces; $H^1(T_X, T_Y, f^*T_Y)$ defined by Namba in [5] and $Ext^i_{\mathcal{C}}((\Omega^1_X, \Omega^1_Y), (\mathcal{O}_X, \mathcal{O}_Y))$ defined by Flenner in [1]. Here $\mathcal{C}$ is an abelian category whose objects are triplets $(\mathcal{F}, \mathcal{G}, \varphi)$, where $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, $\mathcal{G}$ a coherent $\mathcal{O}_Y$-module and $\varphi \in Hom_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$, and for $(\mathcal{F}, \mathcal{G}, \varphi)$, $(\mathcal{F}', \mathcal{G}', \varphi') \in \mathcal{C}$, a morphism from $(\mathcal{F}, \mathcal{G}, \varphi)$ to $(\mathcal{F}', \mathcal{G}', \varphi')$ is $(\alpha, \beta) \in Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}') \times Hom_{\mathcal{O}_Y}(\mathcal{G}, \mathcal{G}')$ such that the diagram

$$\begin{tikzcd} f^*\mathcal{G} \arrow[r, "f^*\beta"] \arrow[d, "\varphi"'] & f^*\mathcal{G}' \arrow[d, "\varphi'"] \\ \mathcal{F} \arrow[r, "\alpha"] & \mathcal{F}' \end{tikzcd}$$

is commutative. The obstruction classes to deformations of $f: X \to Y$ with Y varied belong to $H^2(T_X, T_Y, f^*T_Y)$.

2.1. Proposition. (i) $\mathcal{N}_{Z/Y} \xrightarrow{\sim} f_*\mathcal{T}_{X/Y}$, *and so there exists an isomorphism* $H^i(Z, \mathcal{N}_{Z/Y}) \xrightarrow[\sim]{f^i} H^i(X, \mathcal{T}_{X/Y})$ *for* $i \geq 0$.

(ii) *There exist isomorphisms*

$$H^i(T_X, T_Y, f^*T_Y) \xleftarrow[\sim]{g^i} H^i(Y, T_Y(\log Z)) \xrightarrow[\sim]{h^i} Ext^i_{\mathcal{C}}((\Omega^1_X, \Omega^1_Y), (\mathcal{O}_X, \mathcal{O}_Y)) \text{ for } i \geq 0.$$

Proof. For the proof of (i) we refer to Proposition (9.1) in [8]. Here we prove (ii). By (2.1), (2.2) and (i) of the proposition, we have the following diagram of exact cohomology sequences;

$$(2.3)\quad \begin{tikzcd} \to H^{i-1}(X, \mathcal{T}_{X/Y}) \arrow[r, "\delta_1"] & H^i(X, T_X) \arrow[r, "Jf"] & H^i(X, f^*T_Y) \arrow[r, "u"] & H^i(X, \mathcal{T}_{X/Y}) \to \\ \to H^{i-1}(Z, \mathcal{N}_{Z/Y}) \arrow[r, "\delta_2"] \arrow[u, "\wr", "f^{i-1}"'] & H^i(Y, T_Y(\log Z)) \arrow[r, "\beta"] \arrow[u, "\alpha"'] & H^i(Y, T_Y) \arrow[r, "v"] \arrow[u, "f^*"'] & H^i(Z, \mathcal{N}_{Z/Y}) \to . \arrow[u, "\wr", "f^i"'] \end{tikzcd}$$

Since $f: X \to Y$ is an immersion outside a two-codimensional subset of X (cf. Corollary (4.2) in [8]), it naturally induces a homomorphism $H^i(Y, T_Y(\log Z)) \longrightarrow H^i(X, T_X)$. This is the map α in (2.3). By (2.3) we have an exact sequence of cohomologies

$$(2.4)\quad \begin{aligned} &\longrightarrow H^{i-1}(X, f^*T_Y) \xrightarrow{\delta} H^i(Y, T_Y(\log Z)) \xrightarrow{\alpha\oplus\beta} H^i(X, T_X) \oplus H^i(Y, T_Y) \\ &\xrightarrow{Jf - f^*} H^i(X, f^*T_Y) \longrightarrow . \end{aligned}$$

Here δ is the composite of the homomorphisms:

$$H^{i-1}(X, f^*T_Y) \xrightarrow{u} H^{i-1}(X, \mathcal{T}_{X/Y}) \xrightarrow{(f^{i-1})^{-1}} H^{i-1}(Z, \mathcal{N}_{Z/Y}) \xrightarrow{\delta_2} H^i(Y, T_Y(\log Z)).$$

On the other hand there are exact sequences of cohomologies;

$$(2.5)\quad \begin{aligned} &\longrightarrow H^{i-1}(X, f^*T_Y) \longrightarrow H^i(T_X, T_Y, f^*T_Y) \longrightarrow H^i(X, T_X) \oplus H^i(Y, T_Y) \\ &\longrightarrow H^i(X, f^*T_Y) \longrightarrow \end{aligned}$$

([5, Proposition (3. 6. 9)] and

$$(2.6)\quad \begin{aligned} &\longrightarrow Ext^{i-1}_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{O}_X) \longrightarrow Ext^i_{\mathcal{C}}((\Omega^1_X, \Omega^1_Y), (\mathcal{O}_X, \mathcal{O}_Y)) \\ &\longrightarrow Ext^i_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \oplus Ext^i_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \longrightarrow Ext^i_{\mathcal{O}_X}(f^*\Omega^1_Y, \mathcal{O}_X) \longrightarrow \end{aligned}$$

([6, (2.2)]). By comparing (2.4) with (2.5) and (2.6), we have the assertion (ii). Q.E.D.

By Miyajima-Namba-Flenner's theorem on the existence of the Kuranishi family of deformations of holomorphic maps ([4, Main theorem]), [5, Theorem (3. 6. 10)], [1, Theorem (8.5)]) and Proposition 2.1, we obtain the following.

2.2. Theorem. *For an analytic subspace Z with a locally stable parametrization of a compact complex manifold Y, there exists an analytic family $\mathcal{F}=(\mathcal{Y}, \mathcal{Z}, \pi, M, 0, \psi)$ of logarithmic deformations of a pair (Y, Z) (resp. $\mathcal{F}=(Y \times M, \mathcal{Z}, \pi, M, 0)$ of locally trivial displacements of Z in Y) such that:*

(i) *the characteristic map $\rho_0: T_0(M) \to H^1(Y, T_Y(\log Z))$ (resp. $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ is injective,*

(ii) *it is complete at any point $t \in M$ (resp. it is maximal at any point $t \in M$), and*

(iii) *it is semi-universal at* 0 *(resp. it is universal at* 0).

Furthermore, if $H^2(Y, T_Y(\log Z))=0$ (resp. $H^1(Z, \mathcal{N}_{Z/Y})=0$), then the parameter space M is non-singular and the characteristic map $\rho_0: T_0(M) \to H^1(Y, T_Y(\log Z)$ (resy. $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$) is bijective.

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