Deformations of Locally Stable Holomorphic Maps
and Locally Trivial Displacements of Analytic
Subvarieties with Ordinary Singularities

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Introduction

1°) Let \( W \) be a compact complex manifold of dimension 3. An irreducible analytic subvariety \( S \) of codimension 1 in \( W \) is called a surface with ordinary singularities if, and only if, for each singular point \( p \) of \( S \), there exists on \( W \) a local coordinate \( (x, y, z) \) with the center \( p \) such that; in a neighborhood of \( p \), the surface is defined by one of the following three equations:

(1) \( yz = 0 \) (double point),
(2) \( xyz = 0 \) (triple point),
(3) \( xy^2 - z^2 = 0 \) (cuspidal point).

It is well known that if we project a non-singular algebraic surface embedded in a suffi-
ciently higher dimensional complex projective space into a three dimensional linear subspace $P^3(C)$ by a \textit{generic} linear projection, then the image is a surface with ordinary singularities in $P^3(C)$ and the source algebraic surface is the normalization of the image. In [10] K. Kodaira studied on \textit{locally trivial} displacements of a surface $S$ with ordinary singularities in a threefold $W$ (cf. Definition (8.1) later), and proved that if $S$ is \textit{semi-regular} in $W$, then there exists an effectively parametrized maximal family of \textit{locally trivial} displacements of $S$ in $W$ whose parameter space is non-singular. In [25] M. Namba proved the existence of the universal family of \textit{locally trivial} displacements of a surface with ordinary singularities.

For a given algebraic surface $S$ with ordinary singularities in a threefold $W$, we denote by $v : X \to S$ the normalization of $S$. Then it is clear that $X$ is non-singular. We set $f = v \circ \gamma : X \to W$, where $\gamma : S' \to W$ denotes the inclusion map. The purposes of this paper are to clarify the relation between locally trivial displacements of $S$ in $W$ and deformations of the map $f = v \circ \gamma : X \to W$ above, and to generalize K. Kodaira's results in [10] and M. Namba's ones in [25] to higher dimensional cases. The key point is that we become aware that the map $f = v \circ \gamma : X \to W$ above is \textit{locally stable} in the sense of J. N. Mather. So our study in this paper is closely related with J. N. Mather's theory of stable $C^\infty$ mappings.

2') If we intend to generalize K. Kodaira's results in [10], or M. Namba's ones in [25] to higher dimensional cases, we need to answer the question: \textit{What are higher dimensional ordinary singularities?}. It is natural to define \textit{higher dimensional ordinary singularities} as singularities of the image $\pi(X) \subset P^n(C)$ of a non-singular algebraic manifold $X$ embedded in a sufficiently higher dimensional projective space $P^n(C)$ by a \textit{generic} linear projection $\pi : P^n(C) \to P^n(C)$ after the case of surfaces.

But it is a problem whether we can classify and describe explicitly the singularities of $\pi(X)$ for a \textit{generic} linear projection $\pi$. Though there has not been yet a complete solution of this problem, J. N. Mather gave a solution in some cases. In [22] he showed that if the pair of positive integers $(n, m)$ belongs to so-called \textit{nice range} (cf. Definition (3.3) below), then the restriction $\pi|_X : X \to P^n(C)$ to $X$ $(n = \dim X)$ of a \textit{generic} linear projection $\pi : P^n(C) \to P^n(C)$ is a \textit{locally stable holomorphic map} (cf. Definition (1.6) below), and in [17] he showed that multi-germs $f : (X, S) \to (Y, q)$ ($q$: a point of $Y$, $S$: a finite subset of $X$ with $f(S) = q$) of locally stable holomorphic maps $f : X \to Y$ between complex manifolds can be classified by certain $C$-algebras. Furthermore he gave in [20] a complete classification of such $C$-algebras for the case where $\dim X < \dim Y$ and the pair $(\dim X, \dim Y)$ of positive integers belongs to the \textit{nice range}.

In view of these J. N. Mather's results, we may define an \textit{analytic subvariety with ordinary singularities} as the image $f(X) \subset Y$ of a proper \textit{locally stable} holomorphic map $f : X \to Y$ between complex manifolds with $\dim X < \dim Y$ ($X$ is not necessary to be connected). For such a holomorphic map $f : X \to Y$, we set $Z := f(X)$. Then we can prove that if we denote by $X \xrightarrow{f} Z \xrightarrow{\gamma} Y$ the canonical factorization of the map $f : X \to Y$, then the map $f' : X \to Z$ is nothing but the normalization of $Z$ (Corollary (4.2), Proposition (4.2)). Hence the above definition of an \textit{analytic subvariety with ordinary singularities} is equivalent to the following (Definition (6.1)):
Let \( Z \) be a pure dimensional analytic subvariety of a complex manifold \( Y \). We denote by \( v : X \to Z \) the normalization of \( Z \), and by \( \iota : Z \to Y \) the inclusion map. Then we say that \( Z \) is an analytic subvariety with ordinary singularities of \( Y \) if \( X \) is non-singular and the composite map \( f := \iota \circ v : X \to Y \) is a locally stable holomorphic map.

Defining higher dimensional ordinary singularities as above, we can prove that the classification of germs \((Y, Z, q) \ (q \in Z \subset Y)\) of ordinary singularities is equivalent to that of multi-germs \(f : (X, S) \to (Y, q) \ (S = f^{-1}(q))\) of locally stable holomorphic maps with \( \dim X < \dim Y \) (Proposition (6.3)). Using the classification of \( C \)-algebras associated to multi-germs of locally stable holomorphic maps in [20], "Normal form theorem for stable germs" in [17], and the characterization of the local stability of a holomorphic map by a certain transversal property in [18], we can give all possible defining equations of ordinary singularities in some cases (cf. Chapter II, §2). But it is generally rather difficult to compute all possible defining equations of ordinary singularities explicitly. However our arguments in this paper do not depend on explicit descriptions of ordinary singularities by local coordinates.

3°) Our main result of this paper is as follows (Theorem (11.1), Proposition (9.1), Proposition (9.3)):

Let \( X \) and \( Y \) be connected compact complex manifolds with \( \dim X < \dim Y \), and let \( f : X \to Y \) be a locally stable holomorphic map from \( X \) into \( Y \). We set \( Z := f(X) \), i.e., an irreducible analytic subvariety with ordinary singularities of \( Y \) by definition. We denote by \( \mathcal{D}(f) \) (resp. \( \mathcal{L}(Z) \)) the category of germs of families of deformations of the map \( f : X \to Y \) (resp. that of germs of families of locally trivial displacements of \( Z \) in \( Y \)). Then there exists naturally a functor \( \mathcal{F} : \mathcal{D}(f) \to \mathcal{L}(Z) \) which give rise to an isomorphism between these two categories. Furthermore, if we denote by \( \mathcal{F}_{X/Y} \) (resp. \( \mathcal{N}_{Z/Y} \)) the sheaf of germs of infinitesimal deformations of the map \( f : X \to Y \) (cf. [8]) (resp. that of germs of infinitesimal locally trivial displacements of \( Z \) in \( Y \) (cf. Definition (8.2))), there exists canonically an isomorphism \( \mathcal{N}_{Z/Y} \cong f_! \mathcal{F}_{X/Y} \) of sheaves; hence an isomorphism \( f^! : H^0(Z, \mathcal{N}_{Z/Y}) \cong H^0(X, \mathcal{F}_{X/Y}) \) of cohomology groups such that for any \( D = (\mathcal{F}, F, \pi, M, 0) \in \text{Ob}(\mathcal{D}(f)) \) (the objects of the category \( \mathcal{D}(f) \)) the following diagram commutes:

\[
\begin{array}{ccc}
T_0(M) & \xrightarrow{\tau_0} & H^0(X, \mathcal{F}_{X/Y}) \\
\downarrow{\sigma_0} & & \downarrow{f^!} \\
H^0(Z, \mathcal{N}_{Z/Y}) & & \\
\end{array}
\]

where \( \tau_0 \) denotes the characteristic map of the family \( D \) (cf. [8]) and \( \sigma_0 \) that of the family \( \mathcal{F}(D)=\{(Y \times M, F(\mathcal{F}), \omega, M, 0) \in \text{Ob}(\mathcal{L}(Z)) \ \omega = \pi_{M,f(\mathcal{F})} : F(\mathcal{F}) \to M \} \) (\( \pi_M \) denotes the projection: \( Y \times M \to M \)) (cf. Definition (8.3)).

From this result it follows that the existence of the universal family of locally trivial displacements of \( Z \) in \( Y \) is equivalent to that of the universal family of deformations of the map \( f : X \to Y \). Well, the existence of the universal family of deformations of a holomorphic map was proved by K. Miyajima ([23], [24]). Therefore we conclude that there exists also the universal family of locally trivial displacements of an analytic sub-
variety $Z$ with ordinary singularities in a complex manifold $Y$. Furthermore, if $H^1(Z, \mathcal{N}_{Z/Y})=0$, the parameter space of this universal family turns out to be non-singular (Proposition (11.1), Proposition (11.2)). Concerning these results, we wish to state two comments:

(a) K. Miyajima and the author introduced in [24] the concept of "logarithmic deformations" of the embedded resolution of the ordinary singularities of a surface in a threefold $W$, or more generally, the concept of "logarithmic deformations" of the pair of a simple normal crossing analytic subset of a complex manifold and a holomorphic map. They showed in that paper the existence of "Kuranishi family" for such "logarithmic deformations", and asserted that the existence of the universal family of locally trivial displacements of a surface with ordinary singularities in a threefold follows from this existence theorem as a by-product. However the result of this paper shows that the concept of "logarithmic deformations" is not necessary to derive the existence of the universal family of locally trivial displacements of a surface with ordinary singularities in a threefold;

(b) K. Kodaira's theorem to the effect that; for a "semi-regular" algebraic surface $S$ with ordinary singularities in a threefold $W$, there exists an effectively parametrized maximal family of locally trivial displacements of $W$ in $Y$ whose parameter space is non-singular; follows from our result in this paper if the ambient manifold $W$ satisfies the condition $H^2(W, \mathcal{O}_W)=0$. Because if $H^2(W, \mathcal{O}_W)=0$, the semi-regularity of $S$ in $W$ is equivalent to the condition $H^1(S, \mathcal{N}_{S/W})=0$. Generally there is a gap between Kodaira's theorem and ours for the case of surfaces.

4) The content of the different chapters is as follows:

Chapter I is mainly devoted to prove that a locally stable holomorphic map is a Thom-Boardman map satisfying condition NC (normal crossing) (cf. Definition (4.4), Definition (4.5)). This fact is well-known in $C^\infty$-category. The key point to prove this is to observe that a local version of Multi-jet Transversality Theorem ([5], chapter II, Theorem (4.13)) also holds in complex analytic category (Theorem (3.1)). From the fact that a locally stable holomorphic map is a Thom-Boardman map satisfying condition NC, we shall derive several results useful for our purpose. As a matter of fact, the property of a locally stable holomorphic map which we need in our arguments is that it is 1-generic (cf. Definition (4.2)) and satisfies condition NC relative to the Thom-Boardman singular loci of order 1 (cf. Definition (4.3), Definition (3.5)). From this it follows that a proper locally stable holomorphic map $f: X \to Y$ with $\dim X < \dim Y$ is a finite map (Cor. (4.1)), and that the singular locus

$$S(f) := \{ x \in X | \dim \ker (df)_x \geq 1 \}$$

of such a map $f$ is an analytic subset of codimension $\geq 2$ of $X$ (Cor. (4.2)). These facts have an essential significance in our study of analytic subvarieties with ordinary singularities. For example, it is based on these facts that if we set $Z := f(X)$ for a proper locally stable holomorphic map $f: X \to Y$ with $\dim X < \dim Y$, and if we denote by $X \to Z \subseteq Y$ the canonical factorization of $f$, then $f': X \to Z$ is nothing but the normalization of $Z$ (Corollary (4.2), Proposition (4.2)).

In the last section of chapter I we shall prove that sufficiently small deformations,
subject to a certain condition, of a locally stable holomorphic map are also locally stable (Theorem (5.1)). This is the case for a locally stable holomorphic map \( f: X \to Y \) with \( X, Y \) compact and \( \dim X < \dim Y \). In the proof of this theorem the characterization of a locally stable holomorphic map by infinitesimal stability (cf. Theorem (2.1) in Chapter I, §2 & Theorem (A) in Appendix) play an essential role. The above theorem (Theorem (5.1)) implies the half part of our main theorem; i.e., there exists canonically a functor \( \mathcal{F}: \mathcal{D}(f) \to \mathcal{L}(Z) \).

In chapter II we shall give the definition of analytic subvarieties with ordinary singularities, and that of a family of locally trivial displacements of an analytic subvariety with ordinary singularities in a compact complex manifold (Definition (8.1)). Furthermore, we shall prove that a characteristic map can be defined for such a family as well as in the case of surfaces (cf. Definition (8.3), Proposition (8.1)).

Chapter III is devoted to prove the remainings of our main results. In §9 we shall prove the assertions concerning the relation between the characteristic maps of families \( D \in \text{Ob}(\mathcal{D}(f)) \) and \( \mathcal{F}(D) \in \text{Ob}(\mathcal{L}(Z)) \) (Proposition (9.3)). The existence of an isomorphism \( \mathcal{N}_{X/Y} \cong f^* \mathcal{T}_{X/Y} \) of sheaves (Proposition (9.1)) follows from the characterization of a locally stable holomorphic map by infinitesimal stability and the fact that the codimension of the singular locus \( S(f) \) of a locally stable holomorphic map \( f: X \to Y \) with \( \dim X < \dim Y \) is not less than two. In §10 we shall prove "Relative normalization theorem" to the effect that we can normalize simultaneously a given family of locally trivial displacements of an analytic subvariety (not necessary to be with ordinary singularities) in a complex manifold, and that the resultant family obtained by a simultaneous normalization is uniquely determined up to isomorphisms (Theorem (10.1)). From this theorem it follows that the functor \( \mathcal{F}: \mathcal{D}(f) \to \mathcal{L}(Z) \) in our main theorem is surjective and give rise to an isomorphism of categories. In the last section of chapter III, we shall give a proof of Main theorem, and derive some concluding results.

In the appendix we shall prove two aspects of the stability of infinitesimally stable multi-germ of a holomorphic map. One is that any unfolding of an infinitesimally stable multi-germ of a holomorphic map is trivial (Theorem (A)), another is a complex analytic local version of the fact that a stable \( C^\infty \) map is equivalent to any \( C^\infty \) map sufficiently close to it (Theorem (B)). These two stabilities, which are well known in \( C^\infty \) category, play essential roles in our study of locally stable holomorphic maps and analytic subvarieties with ordinary singularities. Using formal power series expansion, we shall give a new proof for complex analytic cases.

5°) Throughout this paper we use the term of an "analytic variety" in the sense of a "reduced complex space". The author expresses his hearty thanks to his colleague Dr. K. Miyajima of Kagoshima University for useful discussions with him during the preparation of this paper. The proof of the existence of formal solutions in Theorem (A) of the appendix is due to Dr. K. Miyajima. The author expresses his hearty thanks also to Dr. A. Fujiki of Kyoto University, who suggested him the ideas of the proof of "Relative normalization theorem" in Chapter III, §10 (Theorem (10, 1)) and showed him another proof of the theorem. The author has referred to it to complete his insufficient original proof.
Chapter I: Locally stable holomorphic maps

§ 1. Definition of locally stable holomorphic maps

Definition (1.1): Let $X$ and $Y$ be complex manifolds. Let $S$ and $T$ be finite subsets of $X$ and $Y$, respectively. A multi-germ $f : (X, S) \rightarrow (Y, T)$ of a holomorphic map at $S$ is an equivalence class of holomorphic maps $g : U \rightarrow Y$ with $g(S) = T$, where $U$ is a neighborhood of $S$ in $X$, and where $g : U \rightarrow Y$ is equivalent to $h : V \rightarrow Y$ if and only if there is a neighborhood $W \subset U \cap V$ of $S$ in $X$ such that $g|_W = h|_W$. If $g$ is a member of the equivalence class $f$, we call $g$ a representative of $f$ and say $f$ is the germ at $S$ of $g$.

Throughout this paper we shall frequently mix up the concepts of a germ and a representative of it.

Definition (1.2): We define a germ of a parametrized family of multi-germs of holomorphic maps as a multi-germ

$$F : (X \times M, S \times m_0) \rightarrow (Y \times M, T \times m_0)$$

of a holomorphic map which preserves m-levels (i.e., representative $\tilde{F}$ of $F$ have the property that $\tilde{F}(X \times m) \subset Y \times m$ for any $m$ in some neighborhood of $m_0$ in $M$), where $M$ is a complex manifold and $m_0$ is an assigned point of $M$.

Definition (1.3): Let $f : (X, S) \rightarrow (Y, T)$ be a multi-germ of a holomorphic map. Then an unfolding of $f$ is a germ of a parametrized family of multi-germs of holomorphic maps:

$$F : (X \times M, S \times m_0) \rightarrow (Y \times M, T \times m_0)$$

such that $F(x, m_0) = (f(x), m_0)$.

Definition (1.4): An unfolding $F : (X \times M, S \times m_0) \rightarrow (Y \times M, T \times m_0)$ of a holomorphic map is trivial if there exist $m$-levels ($m \in M$) preserving germs of analytic automorphisms $G : (X \times M, S \times m_0) \rightarrow (X \times M, S \times m_0)$ and $H : (Y \times M, T \times m_0) \rightarrow (Y \times M, T \times m_0)$ with $G|_{X \times m_0} = 1_X$, $H|_{Y \times m_0} = 1_Y$, such that $H \circ F \circ G^{-1} = f \times 1_M$.

Definition (1.5): We say a multi-germ $f : (X, S) \rightarrow (Y, T)$ of a holomorphic map is simultaneously stable if any unfolding of $f$ is trivial.

Definition (1.6): Let $f : X \rightarrow Y$ be a holomorphic map between complex manifolds. We say $f$ is locally stable if a multi-germ $f : (X, S) \rightarrow (Y, f(S))$ is simultaneously stable for any finite subset $S$ of $X$.

Remark (1.1): The above definition is equivalent to that a multi-germ $f : (X, S) \rightarrow (Y, q)$ is locally stable for any finite subset $S$ of $X$ with $f(S) = q$, a point.

Remark (1.2): In differential topology the notion of a locally stable $C^\infty$ map is equivalent to that of a stable one if the source manifold is compact. But in complex analytic category, this is no longer the case. (For any $C^\infty$ manifolds $X$ and $Y$, we denote by $C^\infty(X, Y)$ the space of all $C^\infty$ maps from $X$ to $Y$ with the so-called "Whitney $C^\infty$ topology". Then we say $f \in C^\infty(X, Y)$ is stable if there exists a neighborhood $N$ of $f$ in $C^\infty(X, Y)$ such that $f$ is equivalent to $g$ for any $g \in N$; i.e., there exist diffeomorphisms $\phi : X \rightarrow X$ and $\psi : Y \rightarrow Y$ such that $\psi \circ f \circ \phi^{-1} = g$.)
§ 2. Local infinitesimal stability

The local stability of a holomorphic map can be characterized by an algebraic condition due to J. N. Mather, so-called local infinitesimal stability. Now we shall explain this fact. Let \( f: X \to Y \) be a holomorphic map between complex manifolds. We denote by \( \Theta_X \) (resp. \( \Theta_Y \)) the sheaf of germs of holomorphic vector fields on \( X \) (resp. \( Y \)), and denote by \( f^*\Theta_Y \) the pull-back of the sheaf \( \Theta_Y \) by \( f \). We denote by \( \Theta_{X,p} \) (resp. \( f^*\Theta_{Y,p} \)) the stalk of \( \Theta_X \) (resp. \( f^*\Theta_Y \)) at \( p \) for a point \( p \) in \( X \), and by \( \Theta_{Y,q} \) the stalk of \( \Theta_Y \) at \( q \) for a point \( q \) in \( Y \). There is a \( \Theta_{X,p} \)-homomorphism \( \Theta_{X,p} \to f^*\Theta_{Y,p} \) defined by the Jacobian map \((df)_p \); and if \( q = f(p) \), there is also a homomorphism \( \omega f: \Theta_{Y,q} \to f^*\Theta_{X,p} \) over \( f^* \) defined by the pull-back of the map \( f \), where \( f^* \) denotes the homomorphism \( \Theta_{Y,q} \to \Theta_{X,p} \) induced by the map \( f \) (if \( A \) is an \( R \)-module, \( B \) an \( S \)-module, and \( \phi: R \to S \) a ring homomorphism, then \( \phi: A \to B \) is a homomorphism over \( \phi \) if \( \phi(\alpha a + \beta b) = \phi(\alpha)\phi(a) + \phi(\beta)\phi(b) \) for all \( \alpha, \beta \in R, a, b \in A \)). More generally, for any finite sets \( S = \{p_1, \ldots, p_s\} \) of distinct points of \( X \) and \( T = \{q_1, \ldots, q_t\} \) of distinct points of \( Y \), we let

\[
\begin{align*}
\Theta_{X,S} &= \Theta_{X,p_1} \times \cdots \times \Theta_{X,p_s}, \\
f^*\Theta_{Y,S} &= f^*\Theta_{Y,p_1} \times \cdots \times f^*\Theta_{Y,p_s}, \quad \text{and} \\
\Theta_{Y,T} &= \Theta_{Y,q_1} \times \cdots \times \Theta_{Y,q_t}
\end{align*}
\]

and so on. If \( T = f(S) \), then the mappings \( f \) and \( \omega f \) just defined above induces a \( \Theta_{X,S} \)-homomorphism \( \Theta_{X,S} \to f^*\Theta_{Y,S} \) and a homomorphism \( \omega f: \Theta_{Y,T} \to f^*\Theta_{Y,S} \) over \( f^* \), where \( f^* \) denotes the homomorphism \( \Theta_{Y,T} \to \Theta_{X,S} \) induced by the map \( f \).

Definition (2.1): We say a multi-germ \( f: (X, S) \to (Y, T) \) of a holomorphic map is simultaneously infinitesimally stable if

\[
if(\Theta_{X,S}) + \omega f(\Theta_{Y,T}) = f^*\Theta_{Y,S}
\]

holds.

Remark (2.1): We denote by \( \hat{\Theta}_{X,p} \) (resp. \( \hat{\Theta}_{X,a} \)) the completion of \( \Theta_{X,p} \) (resp. \( \Theta_{X,a} \)) in the \( m_p \)-adic (resp. \( m_a \)-adic) topology, where \( m_p \) (resp. \( m_a \)) denotes the maximal ideal of \( \Theta_{X,p} \) (resp. \( \Theta_{X,a} \)). The mappings \( if: \hat{\Theta}_{X,S} \to f^*\hat{\Theta}_{Y,S} \) and \( \omega f: \hat{\Theta}_{Y,T} \to f^*\hat{\Theta}_{Y,S} \) are also defined by the same way as above. Then we can show that the condition \( if(\Theta_{X,S}) + \omega f(\Theta_{Y,T}) = f^*\Theta_{Y,S} \) is equivalent to the one \( if(\hat{\Theta}_{X,S}) + \omega f(\hat{\Theta}_{Y,T}) = f^*\hat{\Theta}_{Y,S} \). In fact J. N. Mather showed that \( if(\Theta_{X,S}) + \omega f(\Theta_{Y,a}) = f^*\Theta_{Y,S} \) (\( T = \{q\}, \) one point) is equivalent to

\[
if(\Theta_{X,S}) + \omega f(\Theta_{Y,a}) + (f^*m_{\phi} + \mathcal{M}_{S}^{\phi}) = f^*\Theta_{Y,S},
\]

hence \( if(\Theta_{X,S}) + \omega f(\Theta_{Y,a}) = f^*\Theta_{Y,S} \) is equivalent to

\[
if(\Theta_{X,S}) + \omega f(\Theta_{Y,a}) + \mathcal{M}_{S}^{\phi} = f^*\Theta_{Y,S},
\]

where \( \mathcal{M}_{S}^{\phi} = \mathcal{M}_{p_1}^{\phi} \times \cdots \times \mathcal{M}_{p_s}^{\phi} \subset \Theta_{X,1} \times \cdots \times \Theta_{X,n} \) and \( m = \dim Y \) (\cite{[16]}, p. 135, Theorem (1.13)). The key points in the proof of this fact are Nakayama's Lemma and Generalized Weierstrass Preparation Theorem (complex analytic version of Theorem (3.6) and Lemma (1.4) at p. 106 and p. 113 in \cite{[5]}). These two facts also hold for finitely generated modules over formal power series ring. Hence by the same argument as J. N. Mather used, we can show that \( if(\Theta_{X,S}) + \omega f(\Theta_{Y,a}) = f^*\Theta_{Y,S} \) is equivalent to

\[
if(\hat{\Theta}_{X,S}) + \omega f(\hat{\Theta}_{Y,a}) + (f^*m_{\phi} + \mathcal{M}_{S}^{\phi}) = f^*\hat{\Theta}_{Y,S},
\]
hence $f(\tilde{\theta}_{x,s}) + \omega f(\tilde{\theta}_{y,t}) = f^* \tilde{\theta}_{y,s}$ is equivalent to

$$f(\tilde{\theta}_{x,s}) + \omega f(\tilde{\theta}_{y,t}) = f^* \tilde{\theta}_{y,s}.$$

It is easy to show that the condition

$$f(\theta_{x,s}) + \omega f(\theta_{y,t}) = f^* \theta_{y,s}$$

is equivalent to

$$f(\tilde{\theta}_{x,s}) + \omega f(\tilde{\theta}_{y,t}) = f^* \tilde{\theta}_{y,s}.$$

Consequently we conclude that $f(\theta_{x,s}) + \omega f(\theta_{y,t}) = f^* \theta_{y,s}$ is equivalent to $f(\tilde{\theta}_{x,s}) + \omega f(\tilde{\theta}_{y,t}) = f^* \tilde{\theta}_{y,s}$, which is the condition adopted by J. N. Mather in [20] to define the notion of infinitesimal stability of a multi-germ of a holomorphic map.

The following theorem is a complex analytic version of a well-known fact in the theory of stable $C^\infty$ mappings.

**Theorem (2.1):** A multi-germ $f : (X, S) \to (Y, T)$ of a holomorphic map is simultaneously stable if and only if it is simultaneously infinitesimally stable.

The proof of ‘only if’ part of this theorem is easy. We will give the proof of ‘if’ part of Theorem (2.1) by using power series expansion in the appendix of this paper (Theorem (A) in Appendix).

**Definition (2.2):** We say a holomorphic map $f : X \to Y$ between complex manifolds is locally infinitesimally stable if a multi-germ $f : (X, S) \to (Y, f(S))$ is simultaneously infinitesimally stable for any finite subset $S$ of $X$.

By Theorem (2.1) the notion of a locally infinitesimally stable holomorphic map is equivalent to that of a locally stable holomorphic map. Hence from now on, we will call “locally stable holomorphic map” for “locally infinitesimally stable holomorphic map” to save breath.

§3. **Local version of Multi-jet Transversality Theorem in complex analytic category**

1°) In $C^\infty$ category Thom’s Transversality Theorem, or more generally, Multi-jet Transversality Theorem (cf. Theorem (4.9) & Theorem (4.13) in [5]) are very powerful tools to study “generic” $C^\infty$ maps. However, in complex analytic category, these theorems do not hold. But a local version is also true in complex analytic category, and we can make use of this fact to study locally stable holomorphic maps. In this section we shall show this fact and derive some results.

We begin by preparing some preliminary propositions. For any complex manifolds $X$ and $Y$ we denote by $Hol(X, Y)$ the set of all holomorphic maps from $X$ into $Y$. For a subset $K$ of $X$ and a subset $W$ of $Y$ we denote by $M(K, W)$ the set $\{ f \in Hol(X, Y) | f(K) \subseteq W \}$. With this notation the family of sets $M(K_1, W_1) \cap ... \cap M(K_m, W_m)$, where $K_1, ..., K_m$ are compact subsets of $X$ and $W_1, ..., W_m$ are open ones of $Y$, form a basis of open subsets for a topology on $Hol(X, Y)$, so-called CO topology (i.e., compact open topology). In the following we denote the set $M(K_1, W_1) \cap ... \cap M(K_m, W_m)$ by $M(K_1, ..., K_m; W_1, ..., W_m)$.

**Definition (3.1):** Let $F$ be a topological space. Then:

(a) a subset $G$ of $F$ is residual if it is the countable intersection of open dense subsets of $F$;
(b) $F$ is a Baire space if every residual set is dense.

**Proposition (3.1):** The space $\text{Hol}(X, Y)$ is a Baire space in CO topology.

**Proof.** The proof is almost identical with the one of Proposition (3.3) in [5]. Let $U_1, U_2, \ldots$ be a countable sequence of open dense subsets of $\text{Hol}(X, Y)$ and $V$ another open subset of $\text{Hol}(X, Y)$ with $V \neq \emptyset$. To prove the proposition it suffices to show that

$$\bigcap_{i=1}^{\infty} U_i \cap V \neq \emptyset.$$ 

Since $V$ is a non-empty open subset of $\text{Hol}(X, Y)$, there exist compact subsets $K_{01}, \ldots, K_{02^n}$ and open subsets $W_{01}, \ldots, W_{02^n}$ of $Y$ such that $M(K_{01}, \ldots, K_{02^n}; W_{01}, \ldots, W_{02^n}) \cap V$ and $M(K_{01}, \ldots, K_{02^n}; W_{01}, \ldots, W_{02^n}) \neq \emptyset$. We may assume that $M(K_{01}, \ldots, K_{02^n}; W_{01}, \ldots, W_{02^n}) \cap \bigcap_{i=1}^{\infty} U_i \neq \emptyset$.

To do this we represent $X = \bigcup_{i=1}^{\infty} X_i$ where each $X_i$ is a relatively compact open subset of $X$ such that $X_i \subset X_{i+1}$ for $i \geq 1$. Along with slight modifications of the arguments used in the proof of Proposition (3.3) in [5], we can choose inductively a sequence of holomorphic maps $f_1, f_2, \ldots$ in $\text{Hol}(X, Y)$; and for each $i$, compact subsets $K_{i1}, \ldots, K_{i\alpha_i}$ of $X$ and open subsets $W_{i1}, \ldots, W_{i\alpha_i}$ of $Y$ satisfying:

\begin{itemize}
  \item[(A)] $f_i \in \bigcap_{j=0}^{\infty} M(K_{j1}, \ldots, K_{j\alpha_j}; W_{j1}, \ldots, W_{j\alpha_j}) \cap U_i$;
  \item[(B)] $M(K_{i1}, \ldots, K_{i\alpha_i}; W_{i1}, \ldots, W_{i\alpha_i}) \subset U_i$;
  \item[(C)] $(i>1)$ $d_{\gamma}(f_i(x), f_{i-1}(x)) < \left(\frac{1}{2}\right)^i$ for all $x \in X_i$;
\end{itemize}

where $d_{\gamma}$ denotes a complete metric on $Y$.

Define $g(x) = \lim_{i \to \infty} f_i(x)$ for $x \in X$. This makes sense since for any $x \in X$ there exists an index $i$ with $x \in X_i$ and the sequence $f_i(x), f_{i+1}(x), \ldots$ is a Cauchy sequence in $Y$ by (C). The condition (C) assures that $f_i(x)$ converges to $g(x)$ uniformly on any compact subset. Therefore $g(x)$ is holomorphic. By the conditions (A) and (B) we can show that

$$g \in M(K_{01}, \ldots, K_{02^n}; W_{01}, \ldots, W_{02^n}) \cap \bigcap_{i=1}^{\infty} U_i.$$ 

This completes the proof of the proposition.

**Proposition (3.2):** Let $X_1, \ldots, X_s, Y_1, \ldots, Y_s$ be complex manifolds ($s \geq 2$). The mapping

$$\delta_s : \text{Hol}(X_1, Y_1) \times \cdots \times \text{Hol}(X_s, Y_s) \longrightarrow \text{Hol}(X_1 \times \cdots \times X_s, \ Y_1 \times \cdots \times Y_s)$$

defined by $(f_1, \ldots, f_s) \mapsto f_1 \times \cdots \times f_s$ is continuous in CO topology.

**Proof.** Since the map $\delta_s$ is factorized through the maps:

$$\text{Hol}(X_1, Y_1) \longrightarrow \text{Hol}(X_1 \times \cdots \times X_s, \ Y_1 \times \cdots \times Y_s) \times \text{Hol}(X_s, Y_s)$$

$$\longrightarrow \text{Hol}(X_1 \times \cdots \times X_s, \ Y_1 \times \cdots \times Y_s)$$

, using induction on $s$, we can reduce the proof to the case $s = 2$. Then by usual arguments, we can prove easily that $\delta_2$ is continuous in CO topology. This completes the proof of the proposition.

2°) For any complex manifolds $X$ and $Y$ we denote by $J^k(X, Y)$ $k$-th jet bundle over $X \times Y$. We may regard $J^1(X, Y)$ as a fibration over $X$ with fibre projection map $\alpha$:
\(J^k(X, Y) \to X\); as a fibration over \(Y\) with fibre projection \(\beta: J^k(X, Y) \to Y\); or as a fibration over \(X \times Y\) with fibre projection map \(\alpha \times \beta: J^k(X, Y) \to X \times Y\). We call \(\alpha: J^k(X, Y) \to X\) the source map and \(\beta: J^k(X, Y) \to Y\) the target map. For a holomorphic map \(f: X \to Y\) from \(X\) into \(Y\), we denote by \(j^k f: X \to J^k(X, Y)\) the \(k\)-th jet extension of \(f\). We denote by \(j^k\) the canonical map: \(\text{Hol}(X, Y) \to \text{Hol}(X, J^k(X, Y))\) given by \(f \mapsto j^k f\). (For precise definition see [5], Chapter II, \S 2.)

**Proposition (3.3):** The map \(j^k: \text{Hol}(X, Y) \to \text{Hol}(X, J^k(X, Y))\) is continuous in CO topology.

**Proof.** Let \(f\) be an element of \(\text{Hol}(X, Y)\). To show that \(j^k\) is continuous at \(f\), it suffices to show that if an open neighborhood of \(j^k f\) in \(\text{Hol}(X, J^k(X, Y))\) which has the form \(M(K, W)\), where \(K\) is a compact subset of \(X\) and \(W\) is an open subset of \(J^k(X, Y)\), is given, then there exists an open neighborhood of \(f\) in \(\text{Hol}(X, Y)\) which has the form \(M(L_1, \ldots, L_l; Z_1, \ldots, Z_l)\) such that

\[j^k(M(L_1, \ldots, L_l; Z_1, \ldots, Z_l)) \subset M(K, W),\]

where \(L_1, \ldots, L_l\) are compact subsets of \(X\) and \(Z_1, \ldots, Z_l\) are open subsets of \(Y\). It is clear that there exists an open neighborhood \(W'\) of \(j^k f(X)\) in \(J^k(X, Y)\) such that \(M(K, W') \subset M(K, W)\). Hence it suffices to show the above under the assumption that \(W\) has the property \(j^k f(X) \subset W\). First, we make open coverings \(\mathcal{U} = \{U_i\}_{i \in I}\), \(\mathcal{U}' = \{U'_i\}_{i \in I}\) of \(X\) and \(\mathcal{V} = \{V_j\}_{j \in J}\) of \(Y\) consisting of domains lying in coordinate neighborhoods such that:

(i) for each \(i \in I\) and \(j \in J\), \(U_i\), \(U'_i\), \(U_i^n\) and \(V_j\) are relatively compact;

(ii) for each \(i \in I\), \(U_i \subset U'_i\) and \(U_i^n \subset U_i^n\);

(iii) for each \(i \in I\), there exists \(j \in J\) such that \(f(U_i^n) \subset V_j\).

For each \(i \in I\) we choose an index \(j \in J\) such that \(f(U_i^n) \subset V_j\) and denote this by \(j(i)\).

For each \(i \in I\) we denote by

\[\phi_i: J^k(X, Y)_{U_i \times V_{j(i)}} \cong U_i^n \times V_{j(i)} \times B_{n,m},\]

the local trivialization of \(J^k(X, Y)\) over \(U_i \times V_{j(i)}\), where \(n = \dim X\), \(m = \dim Y\) and \(B_{n,m} = \bigotimes_{i=1}^m A^n_k\) with \(A^n_k\) the vector space of polynomials in \(n\) variables of degree \(k\) whose constant term equal to zero. Let \(W_i\) be an open neighborhood of \(j^k f(U_i)\) in \(J^k(X, Y)\) such that \(W_i \subset W\) for each \(i \in I\). For a positive number \(\varepsilon\), we set

\[W_{i,\varepsilon} = \{(x, y, p) \in U_i \times V_{j(i)} \times B_{n,m} : \text{d}_{U_i \times V_{j(i)} \times B_{n,m}}(\phi_i(j^k f(x)), (x, y, p)) < \varepsilon\},\]

where \(\text{d}_{U_i \times V_{j(i)} \times B_{n,m}}\) denotes the Euclidean metric on \(U_i \times V_{j(i)} \times B_{n,m}\). We choose a positive number \(\varepsilon\) such that \(\phi_i^{-1}(W_{i,\varepsilon}) \subset W_i\) and denote it by \(\varepsilon_i\). We claim that there exists an open neighborhood \(N_i\) of \(j^k f(U_i)\) in \(J^k(X, Y)_{U_i \times V_{j(i)}} \cong U_i^n \times V_{j(i)}\) with the property that if \(g \in M(U_i, V_{j(i)})\) satisfies \(j^k g(U_i) \subset N_i\), then \(j^k g(U_i) \subset \phi_i^{-1}(W_{i,\varepsilon_i})\). Indeed, for any \(g \in M(U_i, V_{j(i)})\) we denote by \(g = (g_1(x), \ldots, g_m(x))\) the representation of the map \(g|_{U_i}: U_i \to V_{j(i)}\) by the local coordinates on \(U_i\) and \(V_{j(i)}\).

We set

\[T^k g(x) = \sum_{1 \leq i_1 < \cdots < i_n \leq k} \frac{\partial^x_{i_1 + \cdots + i_n} g(x)}{\partial x_{i_1} \cdots \partial x_{i_n}} (x) \cdot X_{i_1} \cdots X_{i_n}\]

for \(x \in U_i\) and \(1 \leq \alpha \leq m\), and \(T^k g(x) = (T^k g_1(x), \ldots, T^k g_m(x))\), where \((x_1, \ldots, x_n)\) is a system of local coordinates on \(U_i\) and \(X_1, \ldots, X_n\) are variables of a polynomial. Then one has
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\[ \phi_i(j^k g(x)) = (x, g(x), T^kg(x)) \in U_i \times V_{f(x)} \times B_{n,m} \]

for any \( x \in U_i \). More explicitly, one has

\[ dU_i \times V_{f(x)} \times B_{n,m} \phi_i(j^k f(x)), \phi_i(j^k g(x))) = dV_{f(x)} \times B_{n,m}((f(x), T^kf(x)), (g(x), T^kg(x))) \]

for any \( x \in U_i \).

Hence, if

\[ \max_{0 \leq v_1 + \cdots + v_n \leq k} \left\{ \left| \frac{\partial^{v_1 + \cdots + v_n} f(x)}{\partial x_1^{v_1} \cdots \partial x_n^{v_n}} - \frac{\partial^{v_1 + \cdots + v_n} g(x)}{\partial x_1^{v_1} \cdots \partial x_n^{v_n}} \right| \right\} < \varepsilon_i \]

we have

\[ d_{U_i \times V_{f(x)} \times B_{n,m}}(\phi_i(j^k f(x)), \phi_i(j^k g(x))) = d_{V_{f(x)} \times B_{n,m}}((f(x), T^kf(x)), (g(x), T^kg(x))) < \varepsilon_i \]

for \( x \in U_i \). This means \( j^k g(U_i) \subset \phi_i^{-1}(W_{i, \delta}) \). Well, using the following lemma (cf. [7], Chapter I, Lemma (5)), we can estimate

\[ \sup_{x \in U_i} \left| \frac{\partial^{v_1 + \cdots + v_n} f(x)}{\partial x_1^{v_1} \cdots \partial x_n^{v_n}} \right| = \sup_{x \in U_i} \left| \frac{\partial^{v_1 + \cdots + v_n} f(x)}{\partial x_1^{v_1} \cdots \partial x_n^{v_n}} \right| \]

by sup \( |f_a - g_a(x)| \).

**Lemma (3.1):** Let \( h \) be a holomorphic function on a bounded domain \( D \) of \( \mathbb{C}^n \) such that

\[ \int_D |h|^2 dw < \infty \]

where \( dw = du_1 du_2 \cdots du_n dv_n \) (\( dx_i = du_i + \sqrt{-1} dv_i \), \( 1 \leq i \leq n \)), and let \( L \) be a compact subset of \( D \). Then one has

\[ \left| \frac{\partial^{v_1 + \cdots + v_n} h}{\partial x_1^{v_1} \cdots \partial x_n^{v_n}} (x) \right| \leq \frac{v_1! \cdots v_n! (v_1 + 2) \cdots (v_n + 2)}{2^n \cdot \pi^{\frac{n}{2}} \cdot \rho_{i=1}^{v_i + \cdots + v_n + n}} \left( \int_D |h|^2 dw \right)^{\frac{1}{2}} \]

for \( x \in L \) where \( \rho \) is the distance between \( L \) and \( \partial D \) (the boundary of \( D \)).

Applying this lemma for \( D = U_i \), \( L = \overline{U}_i \), we have

\[ \sup_{x \in U_i} \left| \frac{\partial^{v_1 + \cdots + v_n} f_a(x)}{\partial x_1^{v_1} \cdots \partial x_n^{v_n}} \right| \leq \frac{v_1! \cdots v_n! (v_1 + 2) \cdots (v_n + 2)}{2^n \cdot \pi^{\frac{n}{2}} \cdot \rho_{i=1}^{v_i + \cdots + v_n + n}} \sup_{x \in U_i} |f_a - g_a(x)| \cdot |Vol(U_i)|, \]

where \( \rho_i = \text{the distance between } \overline{U}_i \text{ and } \partial U_i \). Therefore, if

\[ \sup_{x \in U_i} |f_a - g_a(x)| \leq \frac{2^n \cdot \pi^{\frac{n}{2}} \cdot \rho_{i=1}^{v_i + \cdots + v_n + n}}{v_1! \cdots v_n! (v_1 + 2) \cdots (v_n + 2) (m + \dim B_{n,m}) \sqrt{Vol(U_i)}} \]

then we have

\[ \sup_{x \in U_i} \left| \frac{\partial^{v_1 + \cdots + v_n} (f_a - g_a)}{\partial x_1^{v_1} \cdots \partial x_n^{v_n}} (x) \right| \leq \frac{\varepsilon_i}{m + \dim B_{n,m}} \]

for any \( v_1, \ldots, v_n \) with \( 0 \leq v_1 + \cdots + v_n \leq k, 0 \leq v_1, \ldots, 0 \leq v_n \); hence by the calculation before, we have

\[ d_{U_i \times V_{f(x)} \times B_{n,m}}(\phi_i(j^k f(x)), \phi_i(j^k g(x))) < \varepsilon_i \]

for \( x \in U_i \), and so \( j^k g(U_i) \subset \phi_i^{-1}(W_{i, \delta}) \). We denote the distance between \( f(U_i) \) and
\[ \partial V_{f(0)} \text{ by } \varepsilon'_k, \text{ and} \]

\[ \min \left\{ \frac{2^n \cdot \pi^\frac{n}{2} \rho_1^{n+1} \cdots \rho_n^{n+1}}{v_1 \cdots v_n (v_1 + 2) \cdots (v_n + 2) (m + \dim B_\varepsilon^{n+k}) \cdot \sqrt{\text{Vol}(U'_i)}} \right\} \]

by \( \varepsilon_k \). We set \( \varepsilon'_i = \min(\varepsilon_i, \varepsilon'_k) \), and define the set

\[ N_i = \{(x, y) \in U'_i \times V_{f(0)} \mid |f_\ast(x) - y| < \varepsilon'_i \} \quad \text{for } 1 \leq i \leq m, \]

where \((y_1, \ldots, y_m)\) denotes the system of local coordinates on \( V_{f(0)} \). Then by the arguments till now, \( N_i \) is an open neighborhood of \( f_\ast(U'_i) \) in \( J^0(X, Y)_{\{U'_i \times V_{f(0)}\}} \approx U'_i \times V_{f(0)} \) with the property that if \( g \in M(U'_i, V_{f(0)}) \) satisfies \( j^0 g(U'_i) \subset N_i \), then \( j^k g(U'_i) \subset \phi_i^{-1}(W'_i, x_i) \subset W_i \subset W \).

Now we go back to the beginning of the proof. Since \( K \subset \bigcup_{i=1}^l U_i \) and \( K \) is compact, we may extract a finite subcover indexed by \( 1, 2, \ldots, l \). For each \( i \) with \( 1 \leq i \leq l \) and for any \( x \in \overline{U_i} \), there exist open neighborhoods \( U_{i,x} \) of \( x \) in \( U_i \) and \( V_{f(0),x} \) of \( f(x) \) in \( V_{f(0)} \) such that:

(i) \( f(U_{i,x}) \subset V_{f(0),x} \);

(ii) \( (U_{i,x} \cap U'_i) \times V_{f(0),x} \subset N_i \).

Since the collection \( \{U_{i,x}\} \) where \( x \) is in \( \overline{U_i} \) is an open covering of \( \overline{U_i} \) and since \( \overline{U_i} \) is compact, we may extract a finite subcover indexed by \( x^{(1)}, \ldots, x^{(k)} \) for each \( i \) with \( 1 \leq i \leq l \). We consider the open subset \( \bigcap_{i=1}^l \bigcap_{s=1}^{k_i} M(U_{i,x^{(s)}}^{(s)}, V_{f(0),x^{(s)}}^{(s)}) \) of \( H^0(X, Y) \) in \( CY \) topology. Then it is clear that \( f \in \bigcap_{i=1}^l \bigcap_{s=1}^{k_i} M(U_{i,x^{(s)}}^{(s)}, V_{f(0),x^{(s)}}^{(s)}) \) and that for any \( g \in \bigcap_{i=1}^l \bigcap_{s=1}^{k_i} M(U_{i,x^{(s)}}^{(s)}, V_{f(0),x^{(s)}}^{(s)}) \) one has \( g \in \bigcap_{i=1}^l M(U_i, V_{f(0)}) \) and \( j^0 g(U'_i) \subset N_i \); hence by the property of \( N_i \), \( j^k g(U'_i) \subset \phi_i^{-1}(W'_i, x_i) \subset W_i \subset W \) for \( 1 \leq i \leq l \). Then, since \( K \subset \bigcup_{i=1}^l U_i \), one has \( j^k g(K) \subset W \) for any \( g \in \bigcap_{i=1}^l \bigcap_{s=1}^{k_i} M(U_{i,x^{(s)}}^{(s)}, V_{f(0),x^{(s)}}^{(s)}) \); i.e. \( j^k(\bigcup_{i=1}^l \bigcap_{s=1}^{k_i} M(U_{i,x^{(s)}}^{(s)}, V_{f(0),x^{(s)}}^{(s)}) \subset M(K, W) \). This shows that \( j^k \) is continuous at \( f \) in \( CY \) topology; and, since \( f \) is arbitrary, \( j^k \) is continuous.

Q.E.D.

3°) To state next two propositions we introduce some notations and terminology.

**Definition (3.2):** Let \( X \) and \( Y \) be complex manifolds and \( f : X \to Y \) be a holomorphic map. Let \( W \) be a submanifold of \( Y \) and \( x \) a point in \( X \).

(i) \( f \) intersects \( W \) transversely at \( x \) (denoted by \( f \nmid W \) at \( x \)) if either

(a) \( f(x) \notin W \), or

(b) \( f(x) \in W \) and \( T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX) \) (we denote by \( T_pM \) the tangent space of \( M \) at \( p \) for a complex manifold \( M \) and a point \( p \) in \( M \)).

(ii) If \( A \) is a subset of \( X \), then \( f \) intersects \( W \) transversely on \( A \) (denoted by \( f \nmid W \) on \( A \)) if \( f \nmid W \) at \( x \) for all \( x \in A \). Furthermore we say that \( f \) intersects \( W \) transversely (denoted by \( f \nmid W \) if \( f \nmid W \) on \( X \).

(iii) If \( B \) is a subset of \( W \) we say that \( f \) intersects \( W \) transversely on \( B \) if \( f \nmid W \) at \( x \) for all \( x \in X \) for which \( f(x) \in B \).

(iv) We say that \( f \) intersects \( W \) transversely on \((A, B)\) (denoted by \( f \nmid W \) on \((A, B)\)) if \( f \nmid W \) at \( x \) for all \( x \in A \) for which \( f(x) \in B \).
Proposition (3.4): Let $X$, $B$, and $Y$ be complex manifolds, and let $W$ be a sub-manifold of $Y$. Let $j: B \rightarrow \text{Hol}(X, Y)$ be a mapping (not necessary to be continuous) and define $\Phi: X \times B \rightarrow Y$ by $\Phi(x, b) = j(b)(x)$. Assume that $\Phi$ is holomorphic and that $\Phi|_W$ is dense in $B$.

Proof. The proof is the same as that of Lemma (4.6) in [5] (p. 53).

Proposition (3.5): Let $X$ and $Y$ be complex manifolds with $W$ a submanifold (not necessary to be closed, i.e. locally closed) of $Y$. Furthermore, let $A$ be a compact subset of $X$ and $B$ a closed subset of $W$ with $B \subset W$. We define the set $T_w(A, B) = \{ f \in \text{Hol}(X, Y) : f|_W \text{ on } (A, B) \}$.

Then $T_w(A, B)$ is an open subset of $\text{Hol}(X, Y)$ in CO topology.

Proof. Define a subset $U$ of $J^1(X, Y)$ as follows: let $\sigma$ be a 1-jet with source $x$ and target $y$ and let $f: X \rightarrow Y$ represent $\sigma$. Then $\sigma \in U$ if and only if either

(i) $y \notin B$, or

(ii) $y \in B$ and $T_y = T_y + (df)_x T_x$.

Then $U$ is an open subset of $J^1(X, Y)$ (cf. Proposition (4.5) in [5]). We note that $f \in \text{Hol}(X, Y)$ belongs to $T_w(A, B)$ if and only if $j^1 f(A) \subset U$. Therefore, if we set $M(A, U) = \{ g \in \text{Hol}(X, J^1(X, Y)) : g(A) \subset U \}$, then we have $T_w(A, B) = (j^1)^{-1}(M(A, U))$ where $j^1$ is the natural map $\text{Hol}(X, Y) \rightarrow \text{Hol}(X, J^1(X, Y))$. Since $M(A, U)$ is an open subset of $\text{Hol}(X, J^1(X, Y))$ in CO topology and $j^1$ is continuous by Proposition (3.3) we conclude $T_w(A, B)$ is open.

Q.E.D.

4°) Let $U_1, \ldots, U_s$ be domains in $C^n$, which are mutually disjoint. We denote by $\bigoplus_{i=1}^s U_i$ the disjoint union of $U_1, \ldots, U_s$. Let $f: \bigoplus_{i=1}^s U_i \rightarrow C^m$ be a holomorphic map, and let $f^{(i)} (1 \leq i \leq s)$ be the restriction $f|_{U_i}: U_i \rightarrow C^m$ of $f$ to $U_i$. In this situation we write $f = \bigoplus_{i=1}^s f^{(i)}$ and say that $f$ is the disjoint sum of $f^{(1)}, \ldots, f^{(s)}$. We note that we can identify $\text{Hol}(\bigoplus_{i=1}^s U_i, C^m)$ with $\text{Hol}(U_1, C^m) \times \cdots \times \text{Hol}(U_s, C^m)$ by the correspondence: $f = \bigoplus_{i=1}^s f^{(i)} \rightarrow f^{(1)}, \ldots, f^{(s)}$.

Theorem (3.1): (local version of Multi-jet Transversality Theorem in complex analytic category) With the notations as above, let $W$ be a submanifold of $J^{k_1}(U_1, C^m) \times \cdots \times J^{k_s}(U_s, C^m)$ ($W$ is not necessary to be closed, i.e. locally closed). Let $T_w = \{ f = \bigoplus_{i=1}^s f^{(i)} \in \text{Hol}(\bigoplus_{i=1}^s U_i, C^m) : j^{k_1} f^{(1)} \times \cdots \times j^{k_s} f^{(s)} : U_1 \times \cdots \times U_s \rightarrow \text{transversal to } W \}$.

Then $T_w$ is a dense subset of $\text{Hol}(\bigoplus_{i=1}^s U_i, C^m)$. Furthermore, it is a residual subset of $\text{Hol}(\bigoplus_{i=1}^s U_i, C^m)$.

Proof. We denote by $B^{k_i}$ the space of polynomial mappings from $C^n$ into $C^m$ of degree $k_i$ for $1 \leq i \leq s$, and let $B = B^{k_1} \times \cdots \times B^{k_s}$. For any element $f = \bigoplus_{i=1}^s f^{(i)}$ in $\text{Hol}(\bigoplus_{i=1}^s U_i, C^m)$ we define the map
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\[ j: B \longrightarrow Hol(U_1 \times \cdots \times U_s, J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m)) \]

by

\[ j(b)(x) = (j^k(f^{(1)} + b^{(1)})(x_1), \ldots, j^k(f^{(s)} + b^{(s)})(x_s)) \]

for \( b = (b^{(1)}, \ldots, b^{(s)}) \in B \) and \( x = (x_1, \ldots, x_s) \in U_1 \times \cdots \times U_s \), and define the map

\[ \Phi: U_1 \times \cdots \times U_s \times B \longrightarrow J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m) \]

by \( \Phi(x, b) = j(b)(x) \) for \( (x, b) \in U_1 \times \cdots \times U_s \times B \). We claim that the map \( \Phi \) is a biholomorphic map. To prove this it is clear that it suffices to prove in the case of \( s = 1 \).

We denote by \( A^k_n \) the vector space of polynomials in \( n \) variables \( X_1, \ldots, X_n \) of degree \( \leq k \) which have their constant terms equal to zero, and let \( B^k_{n,m} = \bigoplus_{i=1}^m A^k_n \). Then we can identify \( J^k(U, C^m) \) with \( U \times C^m \times B^k_{n,m} \). Under this identification the map

\[ \Phi: U \times B^k \longrightarrow J^k(U, C^m) = U \times C^m \times B^k_{n,m} \]

is given by \( \Phi(x, b) = (x, (f + b)(x), T_k(f + b)(x)) \) for \( (x, b) \in U \times B^k \), where if we set

\[ T_k(f^{(i)} + b_i)(x) = \sum_{1 \leq i_1 + \cdots + i_n \leq k} \frac{\partial^{i_1 + \cdots + i_n}(f^{(i)} + b_i)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}(x) \cdot X_1^{i_1} \cdots X_n^{i_n} \]

(\( 1 \leq i \leq m \)) for \( f = (f_1, \ldots, f_m): U \rightarrow C^m \) and \( b = (b_1, \ldots, b_m): C^n \rightarrow C^m \), then \( T_k(f + b)(x) \) denotes \( (T_k(f_1 + b_1)(x), \ldots, T_k(f_m + b_m)(x)) \in B^k_{n,m} = \bigoplus_{i=1}^m A^k_n \). It is easy to observe that this correspondence is a biholomorphic map.

Now we go back to the situation at the beginning. Since \( \Phi \) is a biholomorphic map, \( \Phi \) is transversal to \( W \). Then by Proposition (3.4), the set \( \{ b \in B | j(b) \not W \} \) is dense in \( B \). So we can choose the sequence \( b_1, b_2, \ldots \) in \( B \) converging to 0 in \( B \) such that \( j(b_\alpha) \not W \) for any \( \alpha \geq 1 \). We denote by \( f + b_\alpha \) the map \( \bigoplus_{i=1}^s (f^{(i)} + b^{(i)}_\alpha) \in Hol(\bigoplus_{i=1}^s U_i, C^m) \)

where \( b = (b^{(1)}_\alpha, \ldots, b^{(s)}_\alpha) \in B = B^{k_1} \times \cdots \times B^{k_s} \). Then since \( j(b_\alpha) \not W \), we have \( f + b_\alpha \in T_W \) for \( \alpha \geq 1 \), and since \( b_1, b_2, \ldots \) converging to 0 in \( B \), we have \( \lim_{a \to \infty} \| f - (f + b_\alpha) \|_K = \lim_{a \to \infty} \| b_\alpha \|_K \)

= 0 for any compact subset \( K \) of \( \bigoplus_{i=1}^s U_i \). (For any \( g \in Hol(\bigoplus_{i=1}^s U_i, C^m) \) and any compact subset \( K \) of \( \bigoplus_{i=1}^s U_i \) we define \( \| g \|_K = \sup_{1 \leq i \leq m, x \in K} | g(x) | \), where \( g = (g_1, \ldots, g_m): \bigoplus_{i=1}^s U_i \rightarrow C^m \). Thus we conclude that \( T_W \) is dense in \( Hol(\bigoplus_{i=1}^s U_i, C^m) \) in \( CO \) topology.

Next we prove that \( T_W \) is the intersection of countable open dense subsets of \( Hol(\bigoplus_{i=1}^s U_i, C^m) \). To do this, we represent each \( U_i \) \( (1 \leq i \leq s) \) (resp. \( W \)) as \( U_i = \bigcup_{r=1}^\infty U_{i,r} \) (resp. \( W = \bigcup_{r=1}^\infty W_{r} \)), where each \( U_{i,r} \) (resp. \( W_{r} \)) for \( r \geq 1 \) is a relatively compact open subset of \( U_i \) (resp. of \( W \)) such that \( U_{i,r} \subset U_{i,r+1} \) (resp. \( W_{r} \subset W_{r+1} \)) for \( r \geq 1 \). We set

\[ T_W = \bigcap_{r=1}^\infty T_W(\bigoplus_{i=1}^s U_{i,r}, W_{r}) \]

for \( r \geq 1 \). Since \( T_W = \bigcap_{r=1}^\infty T_W(\bigoplus_{i=1}^s U_{i,r}, W_{r}) \), it suffices to show that each \( T_W(\bigoplus_{i=1}^s U_{i,r}, W_{r}) \) is an open dense subset of \( Hol(\bigoplus_{i=1}^s U_i, C^m) \). It is trivial that \( T_W(\bigoplus_{i=1}^s U_{i,r}, W_{r}) \) is dense,
for $T_{w} \subset T_{w}(\bigoplus_{i=1}^{s} U_{i,r}, \overline{W})$ and $T_{w}$ is dense as showed before. To show that $T_{w}(\bigoplus_{i=1}^{s} U_{i,r}, \overline{W})$ is open subset of $\text{Hol}(\bigoplus_{i=1}^{s} U_{i}, C^{m})$ we consider the mapping

$$A: \text{Hol}(\bigoplus_{i=1}^{s} U_{i}, C^{m}) \longrightarrow \text{Hol}(U_{1} \times \cdots \times U_{s}, J^{k_{1}}(U_{1}, C^{m}) \times \cdots \times J^{k_{s}}(U_{s}, C^{m}))$$

defined by $A(\bigoplus_{i=1}^{s} f^{(i)}) = j^{k_{1}}f^{(1)} \times \cdots \times j^{k_{s}}f^{(s)}$ for $\bigoplus_{i=1}^{s} f^{(i)} \in \text{Hol}(\bigoplus_{i=1}^{s} U_{i}, C^{m})$. We set

$$T_{w}(\overline{U}_{1,r} \times \cdots \times \overline{U}_{s,r}, \overline{W}) = \{g \in \text{Hol}(U_{1} \times \cdots \times U_{s}, J^{k_{1}}(U_{1}, C^{m}) \times \cdots \times J^{k_{s}}(U_{s}, C^{m})$$

$$|g|_{\infty}W \text{ on } (\overline{U}_{1,r} \times \cdots \times \overline{U}_{s,r}, \overline{W})\},$$

then by Proposition (3.5), it is an open subset. It is clear that $T_{w}(\bigoplus_{i=1}^{s} U_{i,r}, \overline{W}) = A^{-1}(T_{w}(\overline{U}_{1,r} \times \cdots \times \overline{U}_{s,r}, \overline{W}))$, and so if we show that $A$ is continuous, then we are done. We observe that the map $A$ is a composition of the mappings

$j = j^{k_{1}} \times \cdots \times j^{k_{s}}: \text{Hol}(\bigoplus_{i=1}^{s} U_{i}, C^{m}) = \text{Hol}(U_{1}, C^{m}) \times \cdots \times \text{Hol}(U_{s}, C^{m})$$

$$\longrightarrow \text{Hol}(U_{1}, J^{k_{1}}(U_{1}, C^{m}) \times \cdots \times \text{Hol}(U_{s}, J^{k_{s}}(U_{s}, C^{m}));$$

and

$\delta_{s}: \text{Hol}(U_{1}, J^{k_{1}}(U_{1}, C^{m}) \times \cdots \times \text{Hol}(U_{s}, J^{k_{s}}(U_{s}, C^{m}))$

$$\longrightarrow \text{Hol}(U_{1} \times \cdots \times U_{s}, J^{k_{1}}(U_{1}, C^{m}) \times \cdots \times J^{k_{s}}(U_{s}, C^{m}))$$

defined by $(g_{1}, \ldots, g_{s}) \rightarrow g_{1} \times \cdots \times g_{s}$. Well, by Proposition (3.3) $j$ is continuous, and by Proposition (3.2) $\delta_{s}$ is so, too. Hence $A$ is continuous. This completes the proof.

(5) The following is the main result in the theory of stable $C^{\infty}$ mappings due to J. N. Mather:

Let $X$ and $Y$ be $C^{\infty}$ manifolds of dimension $n$ and $m$, respectively. Then the set of proper stable $C^{\infty}$ maps from $X$ into $Y$ is dense with respect to the Whitney $C^{\infty}$ topology in the set $C_{\text{prop}}^{\infty}(X, Y)$ of proper $C^{\infty}$ maps from $X$ into $Y$ if, and only if, the pair $(n, m)$ satisfies one of the following conditions:

(i) $n < \frac{6}{7}m + \frac{8}{7}$ and $m - n \geq 4$;

(ii) $n < \frac{6}{7}m + \frac{9}{7}$ and $3 \geq m - n \geq 0$;

(iii) $m < 8$ and $m - n = -1$;

(iv) $m < 6$ and $m - n = -2$;

(v) $m < 7$ and $m - n \leq -3$;

provided that $C_{\text{prop}}^{\infty}(X, Y) \neq \emptyset$ ([18], [19]).

Definition (3.3): We say a pair $(n, m)$ of positive integers belongs to "nice range" if it satisfies one of the above conditions.

A local version of the above J. N. Mather's results also holds in complex analytic category. We shall explain this fact in this paragrap. The simultaneous local stability of a multi-germ of a holomorphic map can be characterized by certain transversal property due to J. N. Mather. Let $X$ and $Y$ be complex manifolds. For any $\sigma \in J^{k}(X, Y)$, we set

$$R(\sigma) = \frac{\phi_{X,p}}{(f^{*}m_{q} + m^{k+1})\phi_{Y,q}},$$

where $f : (X, p) \rightarrow (Y, q)$ is any representative of $\sigma$ and $m_{p}$ (resp. $m_{q}$) denotes the maximal
ideal of $\sigma_{x,p}$ (resp. $\sigma_{y,q}$). Clearly $R(\sigma)$ is independent of the choice of $f$. Given $\sigma$ and $\sigma' \in J^k(X, Y)$, we will say $\sigma$ is contact equivalent to $\sigma'$ if $R(\sigma) \simeq R(\sigma')$, where $\simeq$ denotes C-algebra isomorphism. By a contact class in $J^k(X, Y)$, we mean an equivalence class of elements of $J^k(X, Y)$ under the relation of contact equivalence. J. N. Mather showed that every contact class are submanifolds of $J^k(X, Y)$ (cf. Lemma (4.2) in [18]). Now we restrict ourselves to a local situation. Let $U_1, \ldots, U_s$ be as in the proceeding paragraph. If $\sigma=(\sigma_1, \ldots, \sigma_s)$ and $\sigma'=(\sigma'_1, \ldots, \sigma'_s)$ are members of $J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m)$, we say $\sigma$ and $\sigma'$ are contact equivalent if the following conditions are satisfied:

(i) $\sigma_i$ is contact equivalent to $\sigma'_i$ for $1 \leq i \leq s$;

(ii) $q_i = q'_i$ if, and only if, $q_i = q'_j$ for $1 \leq i < j \leq s$ where $q_i$ (resp. $q'_i$) denotes the target of $\sigma_i$ (resp. $\sigma'_i$) (i.e. $q_i = \beta_i(\sigma)$ where $\beta_i : J^k(U_i, C^m) \to C^m$ is the natural projection).

By a contact class in $J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m)$, we mean an equivalence class of elements of $J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m)$ under the relation of contact equivalence. For any $\sigma=(\sigma_1, \ldots, \sigma_s) \in J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m)$, we denote by $C_\sigma$ the contact class in $J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m)$ containing $\sigma$.

**Theorem (3.2):** (J. N. Mather; a local version of the condition (d) at Theorem (4.1) in [18]) Let $f=\bigoplus_{i=1}^s f^{(i)} : \bigoplus_{i=1}^s U_i \to C^m$ be a holomorphic map, and $S=\{p_1, \ldots, p_s\}$ a finite subset of $\bigoplus_{i=1}^s U_i$ with $p_i \in U_i$ for $1 \leq i \leq s$. We set

$$\sigma=(j^k f^{(1)}(p_1), \ldots, j^k f^{(s)}(p_s)) \in J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m).$$

Let $k_i \leq m$ for $1 \leq i \leq s$. Then $f$ is simultaneously locally stable at $S$ if, and only if, the map

$$j^k f^{(1)} \times \cdots \times j^k f^{(s)} : U_1 \times \cdots \times U_s \to J^k(U_1, C^m) \times \cdots \times J^k(U_s, C^m)$$

intersects $C_\sigma$ transversely at $(p_1, \ldots, p_s) \in U_1 \times \cdots \times U_s$.

By Theorem (3.1) and Theorem (3.2) we obtain the following:

**Proposition (3.6):** We define $S.St(\bigoplus_{i=1}^s U_i, C^m)$ to be the subset of $\text{Hol}(\bigoplus_{i=1}^s U_i, C^m)$ consisting of all $f=\bigoplus_{i=1}^s f^{(i)} \in \text{Hol}(\bigoplus_{i=1}^s U_i, C^m)$ which are simultaneously locally stable at any finite subset $S=\{p_1, \ldots, p_s\}$ of $\bigoplus_{i=1}^s U_i$ with $p_i \in U_i$ for $1 \leq i \leq s$. Then $S.St(\bigoplus_{i=1}^s U_i, C^m)$ is a residual subset of $\text{Hol}(\bigoplus_{i=1}^s U_i, C^m)$ if, and only if, the pair $(n, m)$ of positive integers belongs to the "nice range".

**Proof:** The proof is almost identical with that of J. N. Mather's original result explained at the beginning of this paragraph. The proof of this proposition is rather easy, for we restrict ourselves only to the local situation. So we omit the proof.

6°) Let $U_1, \ldots, U_s$ be as in the proceeding paragraph. Let $V_i$ be a submanifold (not necessary to be closed, i.e., locally closed) of $U_i$ for $1 \leq i \leq s$. We allow the case where $V_i = \emptyset$ for some $i$. Let $f$ be a holomorphic map from $\bigoplus_{i=1}^s U_i$ into $C^m$.

**Definition (3.4):** Let $(p_1, \ldots, p_s) \in V_1 \times \cdots \times V_s$ be a point with $f(p_1) = \cdots = f(p_s) = q$, a point in $C^m$. We say $f$ satisfies condition NC (normal crossing condition) at $(p_1, \ldots, p_s)$ relative to $V_1, \ldots, V_s$ if $(df)_{p_1}(T_{p_1} V_1) = \cdots = (df)_{p_s}(T_{p_s} V_s)$ are in a general position in $T_q(C^m)$ in the sense that
\[ \text{codim} \{(df)_{p_1}(T_{p_1}V_1) \cap \cdots \cap (df)_{p_s}(T_{p_s}V_s)\} = \sum_{i=1}^{s} \text{codim} (df)_{p_i}(T_{p_i}V_i). \]

**Definition (3.5):** We say \( f \) satisfies condition NC relative to \( V_1, \ldots, V_s \) if, for any integer \( r \) with \( 1 \leq r \leq s \), any sequence of integers \( i_1, i_2, \ldots, i_r \) with \( 1 \leq i_1 < i_2 < \cdots < i_r \leq s \), and for any point \( (p_{i_1}, \ldots, p_{i_r}) \in V_{i_1} \times \cdots \times V_{i_r} \) with \( f(p_{i_1}) = \cdots = f(p_{i_r}) = q \), one point in \( C^m, f_{i_1} \otimes \cdots \otimes f_{i_r} \) satisfies condition NC at \( (p_{i_1}, \ldots, p_{i_r}) \) relative to \( V_{i_1}, \ldots, V_{i_r} \).

With these terminologies we prove the following lemma.

**Lemma (3.2):** For each \( i = 1, \ldots, s \), let \( W_i \) be a submanifold of \( J^k(U_i, C^m) \) such that the target map \( \beta_{|W_i}: W_i \to C^m \) is a submersion. Let \( f = \bigotimes_{i=1}^s f^{(i)} \) be the element of \( \text{Hol}(U, C^m) \) satisfying the following:

(i) \( j^k f^{(i)} \) for \( 1 \leq i \leq s; \)

(ii) \( \{(j^k f^{(1)} \cdots j^k f^{(s)}) (U_1 \times \cdots \times U_s) \cap \{(W_1 \times \cdots \times W_s) \cap \beta^{-1}(AC^m)\} = \emptyset; \)

where \( AC^m = \{(y_1, \ldots, y_s) \in C^m \times C^m | y_1 = \cdots = y_2\} \) and \( \beta_1 = j^k f^{(1)} \cdots j^k f^{(s)} \times j^k f^{(1)} \cdots j^k f^{(s)} \).

Then \( W \cap \beta^{-1}(AC^m) \) is a non-empty submanifold of \( U_i \) for \( 1 \leq i \leq s \). (The conditions (i) and (ii) assure that each \( W_i \) is a non-empty submanifold of \( U_i \) for \( 1 \leq i \leq s \).)

Then for any point \( (p_1, \ldots, p_s) \in W_1 \times \cdots \times W_s \) with \( f^{(1)}(p_1) = \cdots = f^{(s)}(p_s) = q \), one point in \( C^m \), the following conditions are mutually equivalent:

(\( x \)) \( j^k f^{(1)} \cdots j^k f^{(s)} \cap \{(W_1 \times \cdots \times W_s) \cap \beta^{-1}(AC^m)\} \) at \( (p_1, \ldots, p_s) \);

(\( \beta \)) \( (df^{(i)})_{p_i}(T_{p_i}W_{f^{(i)}}) \cap \beta^{-1}(AC^m) \) are in a general position in \( T_i C^m \).

(By the assumption that the target map \( \beta_{|W_i}: W_i \to C^m \) is a submersion for \( 1 \leq i \leq s \), the target map \( \beta_{|W_i \times \cdots \times W_s}: W_1 \times \cdots \times W_s \to C^m \times \cdots \times C^m \) (s-timess) is also a submersion; hence \( W_1 \times \cdots \times W_s \cap \beta^{-1}(AC^m) \) is a submanifold of \( W_1 \times \cdots \times W_s \).

**Proof:** We set \( W = W_1 \times \cdots \times W_s \) and \( W(f) = W_1(f^{(1)}) \times \cdots \times W_s(f^{(s)}) \). By the condition (i) above, \( j^k f^{(1)} \cdots j^k f^{(s)} \cap W \); hence \( j^k f^{(1)} \cdots j^k f^{(s)} \cap \{W \cap \beta^{-1}(AC^m)\} \) at \( (p_1, \ldots, p_s) \) if and only if \( j^k f^{(1)} \cdots j^k f^{(s)} \cap \{W \cap \beta^{-1}(AC^m)\} \) at \( (p_1, \ldots, p_s) \) where \( j^k f^{(1)} \cdots j^k f^{(s)} \cap \{W \cap \beta^{-1}(AC^m)\} \) denotes the restriction of \( j^k f^{(1)} \cdots j^k f^{(s)} \) to \( W(f) \). We consider the following commutative diagram:

Well, since \( \beta_{|W}: W \to C^m \times \cdots \times C^m \) (s-timess) is a submersion, \( j^k f^{(1)} \cdots j^k f^{(s)} \cap \{W \cap \beta^{-1}(AC^m)\} \) at \( (p_1, \ldots, p_s) \) if and only if \( f^{(1)} \times \cdots \times f^{(s)} \cap \{W \cap \beta^{-1}(AC^m)\} \) at \( (p_1, \ldots, p_s) \). It is easy to see that the condition \( f^{(1)} \times \cdots \times f^{(s)} \cap \{W \cap \beta^{-1}(AC^m)\} \) at \( (p_1, \ldots, p_s) \) is equivalent to that \( (df^{(1)})_{p_1}(T_{p_1}W(f^{(1)})) \cdots, (df^{(s)})_{p_s}(T_{p_s}W(f^{(s)})) \) are in a general position in \( T_i C^m \). Therefore we conclude the conditions (\( x \)) and (\( \beta \)) are mutually equivalent.

Q.E.D.
By Theorem (3.1) and Lemma (3.2) we obtain the following.

**Proposition (3.7):** For each $i=1, \ldots, s$, let $W_i$ be a submanifold of $J^k(U_i, C^n)$ such that the target map $\beta_{W_i}: W_i \to C^n$ is a submersion. We define $T_{NC}(W_1, \ldots, W_s)$ to be the subset of $\text{Hol}(\bigoplus_{i=1}^s U_i, C^n)$ consisting of all $f = \bigoplus_{i=1}^s f^{(i)} \in \text{Hol}(\bigoplus_{i=1}^s U_i, C^n)$ which satisfy the following:

(i) $j^k f^{(i)} \vert_{W_i}$ for $1 \leq i \leq s$;

(ii) if we set $W_i(f^{(i)}) = (j^k f^{(i)})^{-1}(W_i)$ for $1 \leq i \leq s$, then $f$ satisfies condition NC relative to $W_i(f^{(i)}), \ldots, W_s(f^{(i)})$. (The condition (i) assures that each $W_i(f^{(i)})$ is a submanifold, possibly empty, of $U_i$ for $1 \leq i \leq s$.)

Then $T_{NC}(W_1, \ldots, W_s)$ is a residual subset of $\text{Hol}(\bigoplus_{i=1}^s U_i, C^n)$.

§4. Holomorphic Thom-Boardman maps satisfying condition NC

1° Let $X$ and $Y$ be complex manifolds of dimension $n$ and $m$, respectively. For any $f \in \text{Hol}(X, Y)$, we say that $f$ has a singularity of type $S_r$ at $x$ in $X$ if $(df)_x$ drops rank by $r$; i.e., if rank $(df)_x = \min(n, m) - r$. Denote by $S_r(f)$ the singular locus of $f$ of type $S_r$. One expects that for a “generic” $f$ the set $S_r(f)$ turns out to be a manifold and one can define $S_{r_1, r_2, \ldots, r_k}(f) = S_r(f) \vert_{S_{r_1}}$; i.e., the set of points where the map $f_{S_{r_1}}: S_r(f) \to Y$ drops rank by $s$, where $s$ is a positive integer with $r + \max(0, \dim X - \dim Y) \geq s$. More generally, one expects that for a “generic” $f$, this process can be continued indefinitely and one defines $S_{r_1, r_2, \ldots, r_k}(f)$ for any sequence of integers $r_1, \ldots, r_k$ such that $r_1 + \max(0, \dim X - \dim Y) \geq r_2 \geq \cdots \geq r_k \geq 0$. This was conjectured by R. Thom and proved by J. N. Boardman ([3]). Specifically what Boardman proved is the following.

**Theorem (4.1):** (Boardman, [3]) For every sequence of integers $r_1, \ldots, r_k$ such that $r_1 + \max(0, \dim X - \dim Y) \geq r_2 \geq \cdots \geq r_k \geq 0$, one can define a fiber subbundle $S_{r_1, \ldots, r_k}$ of $J^k(X, Y)$ (relative to the fibration $J^k(X, Y) \to X \times Y$) such that, if $j^k f$ is transversal to all manifolds $S_{r_1, \ldots, r_k}$ where $l \leq k$, then $S_{r_1, \ldots, r_k}(f)$ is well-defined and $x \in S_{r_1, \ldots, r_k}(f) \iff j^k f(x) \in S_{r_1, \ldots, r_k}$.

**Remark (4.1):** What Boardman proved is in the case where $X$ and $Y$ are diffeomorphic manifolds. But it is also valid for complex manifolds (See [21]).

**Definition (4.1):** We call the subbundle $S_{r_1, \ldots, r_k}$ of $J^k(X, Y)$ in the above theorem Thom-Boardman singularity of order $k$ whose Boardman symbol is $(r_1, \ldots, r_k)$.

**Definition (4.2):** We say that $f \in \text{Hol}(X, Y)$ is $k$-generic if $j^k f$ intersects transversely every Thom-Boardman singularity of order $k$.

**Definition (4.3):** Let a holomorphic map $f: X \to Y$ be $l$-generic for any positive integer $l$ with $1 \leq k$. Then we call $S_{r_1, \ldots, r_k}(f) = (j^k f)^{-1}(S_{r_1, \ldots, r_k})$ the Thom-Boardman singular locus of $f$ of order $k$ whose Boardman symbol is $(r_1, \ldots, r_k)$.

**Definition (4.4):** We say that $f \in \text{Hol}(X, Y)$ is a Thom-Boardman map if $f$ is $k$-generic for any positive integer $k$.

**Definition (4.5):** Let $f: X \to Y$ be a Thom-Boardman map. We say $f$ satisfies condition NC if for any sequence of Boardman symbols $I_1, \ldots, I_s$ (not necessarily distinct), $f$ satisfies condition NC relative to $S_{r_1}(f), \ldots, S_{r_k}(f)$ (for the definition of “condition NC” see §3).
2°) In \( C^\infty \) category, by Multi-jet Transversality Theorem, we can prove that for any \( C^\infty \) manifolds \( X, Y \), the set of Thom-Boardman maps satisfying the condition NC is dense in the space \( C^\infty(X, Y) \) of all \( C^\infty \) maps from \( X \) to \( Y \) with the Whitney \( C^\infty \) topology. However, this is no longer the case in complex analytic category, because Multi-jet Transversality Theorem does not hold in complex analytic category. But local version is also true. In this paragraph we will show this fact and derive from this that a locally stable holomorphic map is a Thom-Boardman map satisfying condition NC.

Let notations be as in §3.

**Proposition (4.1):** Let \( U_1, \ldots, U_s \) be domains in \( C^n \), which are mutually disjoint. We define \( TB_{NC}(\bigoplus_{i=1}^s U_i, C^m) \) to be the subset of \( Hol(\bigoplus_{i=1}^s U_i, C^m) \) consisting of all \( f = \bigoplus_{i=1}^s f^{(i)} \in Hol(\bigoplus_{i=1}^s U_i, C^m) \) which satisfy the following:

(i) \( f^{(i)} \) is a Thom-Boardman map for \( 1 \leq i \leq s \);

(ii) for any \( s \) sequences of Thom-Boardman symbols \( I_1, \ldots, I_s \) (not necessarily distinct), \( f = \bigoplus_{i=1}^s f^{(i)} \) satisfies the condition NC relative to \( S_{I_1}(f^{(1)}), \ldots, S_{I_s}(f^{(s)}) \) (cf. Definition (3.5) in §3).

Then \( TB_{NC}(\bigoplus_{i=1}^s U_i, C^m) \) is a residual subset of \( Hol(\bigoplus_{i=1}^s U_i, C^m) \).

**Proof:** This is a corollary to Proposition (3.7). Applying Proposition (3.7) for \( S_{I_1} = W_1, \ldots, S_{I_s} = W_s \), we conclude that \( T_{NC}(S_{I_1}, \ldots, S_{I_s}) \) is a residual subset of \( Hol(\bigoplus_{i=1}^s U_i, C^m) \). It is clear that \( TB_{NC}(\bigoplus_{i=1}^s U_i, C^m) = \cap_{(I_1, \ldots, I_s)} T_{NC}(S_{I_1}, \ldots, S_{I_s}) \) where \( (I_1, \ldots, I_s) \) ranges over all \( s \) sequences of Thom-Boardman symbols. Therefore \( TB_{NC}(\bigoplus_{i=1}^s U_i, C^m) \) is residual.

Q.E.D.

**Theorem (4.2):** A locally stable holomorphic map is a Thom-Boardman map satisfying condition NC.

**Proof:** Let \( f: X \to Y \) be a locally stable holomorphic map between complex manifolds. Let \( p_1, \ldots, p_s \) be distinct points in \( X \) with \( f(p_1) = \cdots = f(p_s) = q \), one point in \( Y \). We choose local coordinate neighborhoods \( U_1, \ldots, U_s \) of \( p_1, \ldots, p_s \) respectively, which are mutually disjoint, and \( V \) of \( g \) with \( f(\bigoplus_{i=1}^s U_i) \subset V \). We regard \( V \) as a domain in \( C^m \). Then we may consider the map \( f|_{\bigoplus_{i=1}^s U_i}: U_1 \to V \) is an element of \( Hol(\bigoplus_{i=1}^s U_i, C^m) \) \( (m = \dim Y) \). Since \( f \) is locally stable, by Theorem (B) in Appendix, there exists an open neighborhood \( N \) of \( f|_{\bigoplus_{i=1}^s U_i} \) in \( Hol(\bigoplus_{i=1}^s U_i, C^m) \) which has the following property:

For any \( g \in N \) there exists a quadruple \( (U' = \bigoplus_{i=1}^s U'_i, W, \phi = \bigoplus_{i=1}^s \phi^{(i)}, \psi) \), where \( U' = \bigoplus_{i=1}^s U'_i \) is an open subset of \( U \) with \( p_i \in U'_i \subset U_i \) for \( 1 \leq i \leq s \), \( W \) an open neighborhood of \( q \) with \( f(U') \subset W \), \( \phi = \bigoplus_{i=1}^s \phi^{(i)} \) an analytic isomorphism from \( U' \) onto \( \phi(U') \subset U \) with \( \phi(U'_i) \subset U'_i \) for \( 1 \leq i \leq s \), and \( \psi \) an analytic isomorphism from \( W \) onto \( \psi(W) \subset C^m \) such that:

\[ f|_W = \psi^{-1} g \circ \phi \] on \( U' \).

We take such a neighborhood \( N \) of \( f \) in \( Hol(\bigoplus_{i=1}^s U_i, C^m) \). Then by Proposition (4.1), there exists a \( g = \bigoplus_{i=1}^s g^{(i)} \in N \cap TB_{NC}(\bigoplus_{i=1}^s U_i, C^m) \). For this \( g \in N \) we take the above quadruple \( (U' = \bigoplus_{i=1}^s U'_i, W, \phi = \bigoplus_{i=1}^s \phi^{(i)}, \psi) \). Let \( I_1, \ldots, I_s \) be any \( s \) sequences of Thom-
Boardman symbols. Then \( j^k g^{(i)} S_{i_1} \subset J^k(U_{i_1}, C^n) \) at \( \phi(p_i) \) for \( 1 \leq i \leq s \), where \( k_i \) denotes the order of Thom-Boardman singularity \( S_{i_1} \); and \( g = \bigoplus_{i=1}^s g^{(i)} \) satisfies the condition NC at \( (\phi_1(p_1), \ldots, \phi_s(p_s)) \) relative to \( S_{i_1}(g^{(1)}), \ldots, S_{i_s}(g^{(s)}) \). We set
\[
\sigma_0 = (j^k f^{(1)}(p_1), \ldots, j^k f^{(s)}(p_s)) \in J^k(U_{i_1}, C^n) \times \cdots \times J^k(U_{i_s}, C^n), \quad \text{and} \quad \tau_0 = (j^k g^{(1)}(\phi^{(1)}(p_1)), \ldots, j^k g^{(s)}(\phi^{(s)}(p_s))).
\]
The pair \((\phi, \psi)\) of local analytic isomorphisms induces an analytic isomorphism from an open neighborhood of \( \sigma_0 \) in \( J^k(U_{i_1}, C^n) \times \cdots \times J^k(U_{i_s}, C^n) \) onto an open neighborhood of \( \tau_0 \) in \( J^k(U_{i_1}, C^n) \times \cdots \times J^k(U_{i_s}, C^n) \) as follows:

Let \( \sigma = (\sigma_1, \ldots, \sigma_s) \) be a point of \( J^k(U_{i_1}, C^n) \times \cdots \times J^k(U_{i_s}, C^n) \) such that \( (\sigma_1, \ldots, \sigma_s) \in \bigcup_{i=1}^s \psi(W) \) for \( 1 \leq i \leq s \), where \( \alpha_i : J^k(U_{i_1}, C^n) \to U_{i_1} \) the source map and \( \beta_i : J^k(U_{i_1}, C^n) \to C^n \) the target map for \( 1 \leq i \leq s \). Let \( h = \bigoplus_{i=1}^s h^{(i)} \) is a multi-germ of a holomorphic map at \( (\alpha_1(\sigma_1), \ldots, \alpha_s(\sigma_s)) \) which represents \( \sigma \). Then the action of \( (\phi, \psi) \) on \( \sigma \) is defined by
\[
(\phi, \psi) \sigma = (j^k f^{(1)}(\psi_{\phi}(h^{(1)})), \ldots, j^k f^{(s)}(\psi_{\phi}(h^{(s)})), j^k g^{(1)}(\phi^{(1)}(\alpha_1(\sigma_1))), \ldots, j^k g^{(s)}(\phi^{(s)}(\alpha_s(\sigma_s)))).
\]

From the fact \( f|_{U'} = \psi_{\phi}^{-1} g \phi \) on \( U' \), it follows that
\[
(j^k f^{(1)} \times \cdots \times j^k f^{(s)})(x_1, \ldots, x_s) = (\phi, \psi)^{-1}(j^k g^{(1)} \times \cdots \times j^k g^{(s)})(\phi^{(1)}(x_1), \ldots, \phi^{(s)}(x_s))
\]
for \( (x_1, \ldots, x_s) \in U_1 \times \cdots \times U_s \). Since a Thom-Boardman singularity \( S_{i_1} \times \cdots \times S_{i_s} \) is invariant under the action of \( (\phi, \psi)^{-1} \), and since \( g \in TC\bigoplus_{i=1}^s U_i, C^n \), we conclude that \( j^k f^{(1)} \bigotimes_{i=1}^s S_{i_1} \) at \( p_i \) for \( 1 \leq i \leq s \), and \( f = \bigoplus_{i=1}^s f^{(i)} \) satisfies the condition NC at \( (p_1, \ldots, p_s) \) relative to \( S_{i_1}(f^{(1)}), \ldots, S_{i_s}(f^{(s)}) \). Well, the points \( p_1, \ldots, p_s \) and Thom-Boardman singularities \( S_{i_1}, \ldots, S_{i_s} \) are arbitrary, so \( f \) is a Thom-Boardman map satisfying condition NC.

Q. E. D.

**Corollary (4.1):** Let \( f : X \to Y \) be a locally stable holomorphic map between complex manifolds. We define the set
\[
\Sigma := \{ p \in X | (df)_p : T_p X \to T_{f(p)} Y \text{ is not surjective} \}.
\]
Then \( f^{-1}(f(p)) \cap \Sigma \) is a finite set for any point \( p \in \Sigma \).

**Proof:** Let \( p_1, \ldots, p_s \) be distinct points of \( f^{-1}(f(p)) \), and let \( p_i \in S_{i_1}(f), \ldots, p_i \in S_{i_s}(f) \) where \( 0 \leq i_1 \leq \min \{(\dim X, \dim Y), \ldots, 0 \leq i_s \leq \min \{(\dim X, \dim Y) \) (in the case \( \dim X \leq \dim Y \)). By Theorem (4.2) \( f \) is a Thom-Boardman map satisfying condition NC; especially, 1-generic and satisfying condition NC relative to Thom-Boardman singular loci \( S_0(f), \ldots, S_s(f) \) of order 1 (\( n = \min \{(\dim X, \dim Y) \) Hence one has
\[
\dim Y \geq \text{codim} \{ \bigotimes_{i=1}^s (df)_{p_i}(T_{p_i}(S_{i_1})) \} \subseteq T_{f(p)} Y
\]
\[
= \sum_{i=1}^s \{ \text{codim} (df)_{p_i}(T_{p_i}(S_{i_1})) \} \subseteq T_{f(p)} Y \geq s.
\]
Therefore \( f \) is a finite map. Q. E. D.

3°) Let \( f : X \to Y \) be a proper holomorphic map between complex manifolds with \( \dim X \neq \dim Y \). We set \( Z := f(X) \). By Remmert's proper mapping theorem, \( Z \) is an
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analytic subvariety of $Y$ and there is a factorization $X \subseteq Z \subseteq Y$ of $f$. We define various sets as follows:

$$
S(f) = \{ p \in X \mid \text{Ker}(df)_p \neq 0 \},
$$

$$
M(f) = \{ p \in X \mid \exists p' \in X \text{ s.t. } p' \neq p \text{ and } f(p') = f(P) \},
$$

$$
S(Z) = \{ q \in Z \mid Z \text{ is not a manifold at } q \},
$$

$$
N(Z) = \{ q \in Z \mid Z \text{ is not normal at } q \}.
$$

With these notations we will prove the following for use later.

**Proposition (4.2):** Let $f : X \to Y$ be as above. Furthermore we assume that $f$ is $1$-generic and satisfying condition NC relative to Thom-Boardman singular loci $S_0(f), \ldots, S_n(f)$ of order $1$ ($n = \dim X$). Then:

(i) $S(f)$ is an analytic subset of $X$ of codimension $\geq 1 + (\dim Y - \dim X)$ (possibly empty),

(ii) $N(Z) = S(Z)$ and $f^{-1}(N(Z)) = S(f) \cup M(f)$ (possibly empty),

(iii) $f' : X \to Z$ is the normalization of $Z$.

**Proof:** (i) The analyticity of the set $S(f)$ is obvious. Note that the collection of sets $\{S_1(f), \ldots, S_n(f)\}$ give a stratification of $S(f)$, and the set $S^\cap(f) := \bigcup_{r \leq k \leq n} S_k(f)$ is an analytic subvariety of $S(f)$ for $1 \leq r \leq n$. Since $f$ is 1-generic, the codimension of $S(f)$ ($1 \leq r \leq n$) in $X$ is equal to that of $S_r$ in $J^r(X, Y)$; i.e., $r^2 + r(\dim Y - \dim X)$. Hence one has $\text{codim}(S(f)) \geq 1 + (\dim Y - \dim X)$.

(ii) We note that $f$ is a finite map because of Corollary (4.1). We show that $f(S(f) \cup M(f)) \subseteq N(Z)$. If this is true, we can prove the assertion (ii). Indeed, since $f_{|X\setminus S(f)\cup M(f)} : X \setminus S(f) \cup M(f) \to Y$ is an embedding and $f(X \setminus S(f) \cup M(f)) \cap f(S(f) \cup M(f)) = \emptyset$, one has $f(X \setminus S(f) \cup M(f)) \subseteq Z \setminus S(Z)$; hence $S(Z) \subseteq f(S(f) \cup M(f))$; and so $Z \subseteq f(S(f) \cup M(f)) \subseteq N(Z) \subseteq Z$. It follows that $S(Z) = f(S(f) \cup M(f)) = N(Z)$. Then one has $f^{-1}(N(Z)) = f^{-1}(f(S(f) \cup M(f))) = S(f) \cup M(f)$, for $f(X \setminus S(f) \cup M(f)) \cap f(S(f) \cup M(f)) = \emptyset$. The proof of the fact $f(S(f) \cup M(f)) \subseteq N(Z)$ is divided into three parts:

(a) $p \in M(f) \implies f(p) \in N(Z)$.

We will show that if $p \in M(f)$, then $Z$ is reducible at $f(p)$. Let $p_1$ be a point of $M(f)$ such that $p_1 \neq p$ and $f(p_1) = f(p)$. Since $f$ is a finite map, there exist open neighborhoods $U$ of $p$ (resp. $U_1$ of $p_1$) in $X$ and $V$ of $f(p) = f(p')$ in $Y$ such that $f_{|U} : U \to V$ (resp. $f_{|U_1} : U_1 \to V$) is a proper map ([4], Lemma (3.2)). If $Z$ is not reducible at $f(q)$, then there exists an open neighborhood $V'$ of $f(q)$ in $V$ such that $f(U) \cap V' = \{ f(U) \cap V' \} \cap S(f)$ is an analytic subset of $X$ of codimension $\geq 1 + (\dim Y - \dim X) \geq 2$, $f(S(f))$ is an analytic subset of $X$ of codimension $\geq 2$. Hence $f(U) \cap V' \subseteq \bigcup \{ f(U) \cap V' \} \setminus S(f))$ is an open dense subset of $f(U) \cap V' \subseteq \bigcup \{ f(U) \cap V' \} \setminus S(f)$.

(b) $p \in S(f) \setminus M(f) \implies f(p) \in N(Z)$.

We take open neighborhoods $U$ of $p$ in $X$ and $V$ of $f(p)$ in $Y$ such that $f_{|U} : U \to V$ is...
a proper map. Taking $U$ sufficiently small, we may assume that $U \cap M(f) = \emptyset$. Then $f_{|U}: U \to V$ is an injection and $f(U) \cap f(X(U)) = \emptyset$. Hence $Z$ is irreducible at $f(p)$ and $f(U) = V \cap Z$. We will show that $f(U \cap S(f)) = S(f(U))$ (the singular locus of $f(U)$).

Take any point $p' \in S(f) \cap U$. If $f(p')$ is a non-singular point of $f(U)$, then there exists an open neighborhood $W$ of $f(p')$ in $f(U)$ which is analytically isomorphic to a domain in a numerical complex affine space. We set $U' = f^{-1}(W)$. Then $f_{|U'}: U' \to W$ is a bijection and the map $f_{|U' \cap S(f)}: U' \cap S(f) \to W \setminus f(S(f))$ is a biholomorphic map. Well, we have codim $S(f) \geq 2$ by the assertion (i) of the proposition. Hence by Riemann's removable singularity theorem, the map $f_{|U' \cap S(f)}: U' \cap S(f) \to W \setminus f(S(f))$ and its inverse can be extended to holomorphic maps on $U'$ and $W$, respectively. This means that the map $f_{|U'}: U' \to W$ is an analytic isomorphism. But this is a contradiction to the fact $p' \in S(f)$.

Hence $f(p') \in S(f(U))$, so $f(U \cap S(f)) \subset S(f(U))$. On the other hand, since $f_{|U' \cap S(f)}: U' \cap S(f) \to V$ is an embedding, one has $f(U \cap S(f)) \subset f(U) \cap S(f(U));$ hence $S(f(U)) \subset f(U \cap S(f))$. Consequently we have $f(U \cap S(f)) = S(f(U))$.

Now we will show that $f(p) \in N(Z)$. We denote by $\partial_{Z,f(p)}$ the stalk at $f(p)$ of the sheaf of germs of weakly holomorphic functions on $Z$, i.e., holomorphic functions on $Z \setminus S(Z)$ and locally bounded at any point in $S(Z)$. There is canonically an injective homomorphism: $\varphi_{U,p} \to \partial_{Z,f(p)}$, because $f_{|U \cap S(f)}: U \setminus S(f) \to f(U) \setminus S(f(U))$ is an analytic isomorphism. Hence the above canonical homomorphism $\varphi_{U,p} \to \partial_{Z,f(p)}$ is surjective. Therefore $\varphi_{U,p} \cong \partial_{Z,f(p)}$. Then, if $f(p)$ is a normal point of $Z$, one has $\partial_{Z,f(p)} = \partial_{Z,f(p)}$, hence $\varphi_{U,p} \cong \partial_{Z,f(p)}$. This is a contradiction, for $f(p)$ is a singular point of $Z$. Consequently $f(p) \in N(Z)$.

(c) $p \in \overline{M(f)} \setminus M(f) \implies f(p) \in N(Z)$:

First we show that there exist sequence $\{p_n\}$ and $\{p'_n\}$ of points in $M(f)$ such that $p_n \neq p'_n$ for any $n$ and $\lim_{n \to \infty} p_n = \lim_{n \to \infty} p'_n = p$, so $p \in S(f)$. Since $p \in \overline{M(f)}$ there exists a sequence $\{p_n\}$ of points in $M(f)$ such that $\lim_{n \to \infty} p_n = p$. Then by the definition of $M(f)$, there exists a sequence $\{p'_n\}$ of points in $M(f)$ such that $p'_n \neq p_n$ and $f(p'_n) = f(p_n)$ for any $n$. By the assumption that $f$ is proper, we may assume that the sequence $\{p'_n\}$ converges to a point. We set $p' = \lim_{n \to \infty} p'_n$. If $p' \neq p$, then $p \in M(f)$ because $f(p) = \lim_{n \to \infty} f(p_n) = \lim_{n \to \infty} f(p'_n) = f(p')$. This is a contradiction to $p \in \overline{M(f)} \setminus M(f)$; hence $p' = p$, so $p \in S(f)$.

Let $f_{|U}: U \to V$ be as in the case (b). $Z$ is irreducible at $f(p)$ for $p \in M(f)$; hence, taking $V$ sufficiently small, one has $f(U) = Z \cap V$. Since $f(p)$ is a concentrated point of $f(M(f))$, $f(U)$ is singular at $f(p)$. Note that $f(U \cap M(f) \cup S(f)) = S(f(U))$ by (a) and (b), and that the map $f_{|U \cap M(f) \cup S(f)}: U \cap M(f) \cup S(f) \to f(U) \setminus S(f(U))$ is an analytic isomorphism. By this notice we infer that any element of $\partial_{X,p}$ defines an element of $\partial_{Z,f(p)}$. If $f(p)$ is a normal point of $Z$, one has $\partial_{Z,f(p)} = \partial_{Z,f(p)}$; hence any element of $\partial_{X,p}$ is the pull-back of an element of $\partial_{Z,f(p)}$ by $f$. From this it follows that for any local holomorphic function $h$ at $p$, there exist an open neighborhood $U'$ of $p$ in $U$ and a natural number $N$ such that $h(p_n) = h(p'_n)$ for $n \geq N$ where $\{p_n\}$ and $\{p'_n\}$ are the sequences of points in $M(f)$ taken above. This is obviously a contradiction. Hence $f(p) \in N(Z)$.

(iii) Since $X$ is non-singular and $f$ is a finite map, the assertion (iii) follows
Corollary (4.2): Let $f: X \to Y$ be a proper locally stable holomorphic map between complex manifolds with $\dim X < \dim Y$. Then the assertions of Proposition (4.2) hold for $f$.

Proof: By Theorem (4.2) $f$ is a Thom-Boardman map satisfying condition $NC$; especially, 1-generic and satisfying condition $NC$ relative to Thom-Boardman singular loci $S_0(f), \ldots, S_n(f)$ ($n = \dim X$) of order 1. Q.E.D.

§ 5. Deformations of locally stable holomorphic maps

1°) Lemma (5.1): Let $f: (X, S) \to (Y, q)$ be a simultaneously stable multi-germ of a holomorphic map (cf. Definition (1.5)). Then $F := f \times \text{id}_M: (X \times M, S \times t_0) \to (Y \times M, q \times t_0)$ is also simultaneously stable, where $M$ is a complex manifold and $t_0$ is an assigned point of $M$.

Proof: By Theorem (2.1) (=Theorem (A) in Appendix) and Remark (2.1) it is enough to show that

$$tF(\partial_{X \times M, S \times t_0}) + \omega F(\partial_{Y \times M, q \times t_0}) = F^* \partial_{Y \times M, S \times t_0}$$

holds. For simplicity we shall show this for the case where $S$ is one point, say $p$. For general case we can prove this by the same way.

We denote $(x_1, \ldots, x_m)$ (resp. $(y_1, \ldots, y_m)$, resp. $(t_1, \ldots, t_l)$) a local coordinate system centered $p$ (resp. at $q$, resp. at $t_0$). Let $\hat{\theta} \in F^* \partial_{Y \times M, p \times t_0}$ be given. We represent it in the form

$$\hat{\theta} = \sum_{j=1}^{m} a_j(x, t) f^* \left( \frac{\partial}{\partial y_j} \right) - \sum_{k=1}^{l} b_k(x, t) \left( \frac{\partial}{\partial t_k} \right)$$

where $a_j(x, t), b_k(x, t) \in \partial_{X \times M, p \times t_0}$ for $1 \leq j \leq m$ and $1 \leq k \leq l$. We expand each $a_j(x, t)$ into a power series in $(t_1, \ldots, t_l)$ with coefficients in $\hat{\partial}_X$ as $a_j(x, t) = \sum_{i_1, \ldots, i_l \geq 0} a_{i_1, \ldots, i_l}(x) t_1^{i_1} \cdots t_l^{i_l}$.

For each multi-index $(v_1, \ldots, v_l)$ we set $\hat{\theta}_{v_1, \ldots, v_l} = \sum_{j=1}^{m} a_{j, v_1, \ldots, v_l}(x) f^* \left( \frac{\partial}{\partial y_j} \right)$. Since $f$ is stable at $p$, there exist $\xi_{v_1, \ldots, v_l} \in \partial_{X, p}$ and $\zeta_{v_1, \ldots, v_l} \in \partial_{Y, q}$ such that $f(\xi_{v_1, \ldots, v_l}) + \omega f(\zeta_{v_1, \ldots, v_l}) = \hat{\theta}_{v_1, \ldots, v_l}$. We set

$$\xi = \sum_{v_1, \ldots, v_l \geq 0} t_1^{i_1} \cdots t_l^{i_l} \xi_{v_1, \ldots, v_l} + \sum_{k=1}^{l} b_k(x, t) \left( \frac{\partial}{\partial t_k} \right)$$

and

$$\zeta = \sum_{v_1, \ldots, v_l \geq 0} t_1^{i_1} \cdots t_l^{i_l} \zeta_{v_1, \ldots, v_l},$$

then $\hat{\xi} \in \partial_{X \times M, p \times t_0}$, $\hat{\zeta} \in \partial_{Y \times M, q \times t_0}$ and $tF(\hat{\xi}) + \omega F(\hat{\zeta}) = \hat{\theta}$ holds. Therefore $F := f \times \text{id}_M: (X \times M, p \times t_0) \to (Y \times M, q \times t_0)$ is a simultaneously stable multi-germ of a holomorphic map.

Q.E.D.

Remark (5.1): By the proof above we note that if $\hat{\theta} \in F^* \partial_{Y \times M, p \times t_0}$ does not involve $\left( \frac{\partial}{\partial t_1} \right), \ldots, \left( \frac{\partial}{\partial t_l} \right)$, then we can take $\xi \in \partial_{X \times M, p \times t_0}$ and $\zeta \in \partial_{Y \times M, q \times t_0}$ so that they also do not involve $\left( \frac{\partial}{\partial t_1} \right), \ldots, \left( \frac{\partial}{\partial t_l} \right)$.

2°) Definition (5.1): Let $f: X \to Y$ be a holomorphic map between complex manifolds. By a family of deformations of the holomorphic map $f: X \to Y$ parametrized by
an analytic variety $M$, we mean a quintet $(\mathcal{X}, F, \pi, M, 0)$ of analytic varieties $\mathcal{X}$, $M$, holomorphic maps $F: \mathcal{X} \to \mathcal{Y} = Y \times M$, $\pi: \mathcal{X} \to M$ and an assigned point $0$ of $M$ with following properties:

(i) $\pi$ is a surjective smooth holomorphic map (not necessary to be proper),
(ii) $\pi_{\mathcal{X}}\circ F = \pi$ where $\pi_{\mathcal{X}}: \mathcal{X} \to M$ is the projection onto the second factor,
(iii) $F_0: X_0 \to Y$ is equivalent to $f: X \to Y$ in the sense that there exists an isomorphism $\phi_0: X \to X_0$ such that $f = F_0 \circ \phi_0$. (We denote $\pi^{-1}(i)$ by $X_i$, and $F_{\mid X_i}: X_i \to Y \times Y_i$ for any point $i \in M$.)

**Theorem (5.1):** Let $(\mathcal{X}, F, \pi, M, 0)$ be a family of deformations of a locally stable holomorphic map $f: X \to Y$ with $Y$ compact ($X$ is not necessary to be compact). We define the set

$$\Sigma := \{ p \in \mathcal{X}' \mid (dF)_p: T_p(\mathcal{X}') \to T_{F(p)}(Y \times M) \text{ is not surjective} \}$$

which is equipped with the structure of a reduced complex space. We assume that $F_{\mid X_0}: \Sigma \to Y \times X_0$ is a proper map. Then there exists an open neighborhood $M'$ of $0$ in $M$ such that for any $t \in M'$ the map $F_t: X_0 \to Y$ is a locally stable holomorphic map.

**Proof:** First, we shall show that there exists an open neighborhood $M''$ of $0$ in $M$ such that $F_{\mid X_0 \cap \pi^{-1}(M'')}: \Sigma \cap \pi^{-1}(M'') \to Y \times M''$ is a finite map. In the following we use a notation

$$R(g) := \mathcal{O}_{Z, p}/(g \cdot m_p) \mathcal{O}_{Z, p}$$

for any holomorphic map $g: Z \to W$ between complex spaces and any point $p$ in $Z$, where $q := g(p) \in W$ and $m_q$ denotes the maximal ideal of $\mathcal{O}_{W, q}$. Since $F_0 = f: X_0 \to Y$ is locally stable, there exist, for any point $p \in \Sigma \cap X_0$, open neighborhoods $\mathcal{U}_p$ of $p$ in $\mathcal{X}$, $\mathcal{V}_{F(p)}$ of $F(p)$ in $Y \times M$, and $t$-level preserving analytic isomorphisms $\phi_p: \mathcal{U}_p \to U_p \times N_p$ and $\psi_p: \mathcal{V}_{F(p)} \to V_p \times N_p$ such that $F_{\mid \mathcal{V}_{F(p)}} = \psi_p^{-1} \circ (f_{\mid U_p} \times id_{N_p}) \circ \phi_p$, where $U_p = \mathcal{U}_p \cap X_0$, $V_p = \mathcal{V}_{F(p)} \cap \{Y \times 0\}$ and $N_p = \pi(\mathcal{U}_p)$. It is obvious that $\phi_p(\mathcal{U}_p \cap \Sigma) = \{U_p \cap \Sigma(f)\} \times N_p$, where $\Sigma(f)$ denotes the set

$$\{p' \in X \mid (dF)_p: T_p X \to T_{F(p)} Y \text{ is not surjective} \}.$$ 

Hence, restricting the above equality to $\mathcal{U}_p \cap \Sigma$, one has $F_{\mid \mathcal{U}_p \cap \Sigma} = \psi_p^{-1} \circ (f_{\mid U_p} \times id_{N_p}) \circ \phi_p$ on $\mathcal{U}_p \cap \Sigma$; and so $R(F_{\mid X_0}) \cong R(f_{\mid X_0})$. Well, by the assumption that $f$ is locally stable and by Corollary (4.1), $f_{\mid X_0}: \Sigma(f) \to Y$ is a finite map. From this it follows that $\dim C R(f_{\mid X_0}) < \infty$ ([4], Theorem (1.11)); hence $\dim C R(F_{\mid X_0}) < \infty$ by the above isomorphism. Then, by the semi-continuity of $\dim C R(F_{\mid X_0})$ concerning $p' \in \Sigma$ ([5], Proposition (2.3)), taking $\mathcal{U}_p$ sufficiently small, one has $\dim C R(F_{\mid X_0}) < \infty$ at any point $p'$ in $\mathcal{U}_p \cap \Sigma$. From now on we assume that this holds on $\mathcal{U}_p$.

Since the collection $\{\mathcal{U}_p\}$, where $p$ in $\Sigma(f) \cap X_0$, is an open covering of $\Sigma(f) = \Sigma \cap X_0$, and since $\Sigma(f) = \Sigma \cap X_0$ is compact, we may extract a finite subcover indexed by $p_1, \ldots, p_k$. By the assumption $F_{\mid X_0}: \Sigma \to Y \times M$ is proper, there exists an open neighborhood $M''$ of $0$ in $M$ such that $\pi^{-1}(M'') \cap \Sigma \subseteq \bigcup_{i=1}^k \mathcal{U}_{p_i}$. Then the way to choose $\mathcal{U}_{p_1}, \ldots, \mathcal{U}_{p_k}$ ensures $\dim C R(F_{\mid X_0}) < \infty$ for any point $p$ in $\pi^{-1}(M'') \cap \Sigma$. Therefore we conclude that $F_{\mid X_0 \cap \pi^{-1}(M'')}: \Sigma \cap \pi^{-1}(M'') \to Y \times M''$ is a finite map ([4], Theorem (1.11)).

We define the sheaves
\[ \Theta_{\mathcal{Y}} := \ker \{ F^* \Theta_{Y \times M} \to (\pi_M^* \circ F)^* \Theta_M \}, \quad \text{and} \]
\[ \Theta_{\mathcal{Y}} := \ker \{ \Theta_{\mathcal{Y}} \to (\pi_M^* \circ F)^* \Theta_M \}, \]
where the homomorphism \( \Theta_{\mathcal{Y}} \to (\pi_M^* \circ F)^* \Theta_M \) is the composition of \( tF \): \( \Theta_{\mathcal{Y}} \to F^* \Theta_{Y \times M} = (\pi_Y^* \circ F)^* \Theta_Y \oplus (\pi_M^* \circ F)^* \Theta_M \) and the canonical projection: \( F^* \Theta_{Y \times M} \to (\pi_M^* \circ F)^* \Theta_M \). The mapping \( tF \): \( \Theta_{\mathcal{Y}} \to F^* \Theta_{Y \times M} \) (cf. \( \S 2 \)) induces naturally the mapping: \( \Theta_{\mathcal{Y}} \to \Theta_{\mathcal{Y}} \) which we denote by the same letter \( tF \). We set \( \mathcal{G} = \Theta_{\mathcal{Y}} / tF(\Theta_{\mathcal{Y}}) \). Now we wish to show that the direct image \( F_* \mathcal{G} \) of the sheaf \( \mathcal{G} \) by \( F \) is a coherent sheaf on \( Y \times M'' \), where \( M'' \) is any relatively compact open neighborhood of \( 0 \) in \( M \) with \( M'' \subseteq M \). We set \( \mathcal{S} := \mathcal{G}(\Sigma) \), the ideal sheaf of the analytic subvariety \( \Sigma \). Note that \( \text{Supp} \mathcal{G} \subseteq \Sigma \). Hence by Rückert’s Nullstellensatz ([6]), there exists a natural number \( N \) such that \( \mathcal{G}^{N} = 0 \) on \( \pi^{-1}(M'') \).

We consider the following sequences of exact sequences of sheaves on \( \pi^{-1}(M'') \):
\[
0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}|_{\Sigma} \longrightarrow 0,
\]
\[
0 \longrightarrow \mathcal{G} \oplus \mathcal{G} \longrightarrow \mathcal{G} \oplus \mathcal{G} \longrightarrow \mathcal{G}|_{\Sigma} \oplus \mathcal{G}|_{\Sigma} \longrightarrow 0,
\]
\[
(5.1)
\]
\[
0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}|_{\Sigma} \oplus \mathcal{G}|_{\Sigma} \longrightarrow 0.
\]

We claim that \( F_* \mathcal{G} \rightarrow (F|_{\Sigma})_* \mathcal{G}|_{\Sigma} \) is surjective for any \( l \) with \( 0 \leq l \leq N - 1 \). Indeed, let \( q \) be any point in \( F(\Sigma) \cap (Y \times M') \), and let \( \{ p_1, \ldots, p_s \} \) be the set of all distinct points of \( F^{-1}(q) \cap \Sigma \). Note that
\[
\{(F|_{\Sigma})_* \mathcal{G}|_{\Sigma} \} \cong \bigoplus_{i=1}^{s} \mathcal{G}|_{\Sigma}.
\]
Choose an open neighborhood \( U_1, \ldots, U_s \) of \( p_1, \ldots, p_s \) in \( \Sigma \), respectively, which are mutually disjoint.

Let
\[ a_q = (b_1, p_1, \ldots, b_s, p_s) \in \{(F|_{\Sigma})_* \mathcal{G}|_{\Sigma} \} \cong \bigoplus_{i=1}^{s} \mathcal{G}|_{\Sigma} \]
be given. Take a cross-section \( a \in \Gamma(V \cap F(\Sigma), (F|_{\Sigma})_* \mathcal{G}|_{\Sigma}) \) which represents \( a_q \) at \( q \), where \( V \) is a Stein open neighborhood of \( q \) in \( Y \times M \). If we take \( V \) sufficiently small, we may assume that \( F^{-1}(V) \cap \Sigma \subseteq \bigcup_{i=1}^{s} U_i \). Let \( b_i \in \Gamma(F^{-1}(V) \cap U_i \cap \Sigma, \mathcal{G}|_{\Sigma}) \) be a cross-section which represents \( b_i, p_i \) at \( p_i \) for \( 1 \leq i \leq s \). Since \( F^{-1}(V) \cap U_i \) is a Stein open subset ([6], p. 33 ~ p. 34), there exists a cross-section \( b_i \in \Gamma(F^{-1}(V) \cap U_i, \mathcal{G}|_{\Sigma}) \) such that \( b_i|_{F^{-1}(V) \cap U_i} \subseteq \Sigma = b_i \) for \( 1 \leq i \leq s \). Well, since \( \text{Supp} \mathcal{G} \subseteq \Sigma \), there exists a cross-section \( b \in \Gamma(F^{-1}(V), \mathcal{G}|_{\Sigma}) \) such that \( b|_{F^{-1}(V) \cap U_i} = b_i \) for \( 1 \leq i \leq s \). We may consider \( b \) to be an element of \( \Gamma(V, F_* \mathcal{G}) \). It is clear \( b = a_q \), and so the sheaf homomorphism \( F_* \mathcal{G} \rightarrow (F|_{\Sigma})_* \mathcal{G}|_{\Sigma} \) is surely surjective. Then taking direct images of the exact sequences of sheaves on \( \pi^{-1}(M'') \) at (5.1), we have the following sequences of exact sequences of sheaves on \( Y \times M'' \):
\[
0 \longrightarrow F_* \mathcal{G} \longrightarrow F_* \mathcal{G} \longrightarrow (F|_{\Sigma})_* \mathcal{G}|_{\Sigma} \longrightarrow 0,
\]
\[
(5.2)
\]
\[
0 \longrightarrow F_* \mathcal{G} \longrightarrow F_* \mathcal{G} \longrightarrow (F|_{\Sigma})_* \mathcal{G}|_{\Sigma} \longrightarrow 0,
\]
\[
0 \longrightarrow F_* \mathcal{G} \longrightarrow F_* \mathcal{G} \longrightarrow (F|_{\Sigma})_* \mathcal{G}|_{\Sigma} \longrightarrow 0.
\]
Note that \((F_{\overline{I}})^{\mathcal{G}}_{\overline{I}}\) is a coherent sheaf on \(Y \times M\) for \(0 \leq I \leq N - 1\), because \(F_{\overline{I}}: \Sigma \to Y \times M\) is proper by assumption. By the last exact sequence at (5.2), we conclude \(F_{\mathcal{G}}^{N-1} \mathcal{G}\) is coherent on \(Y \times M'\). Then, by the last exact sequence but one at (5.2), we conclude \(F_{\mathcal{G}}^{N-2} \mathcal{G}\) is coherent on \(Y \times M''\). Continuing this argument successively, we conclude \(F_{\mathcal{G}}\) is coherent on \(Y \times M''\) as desired.

Finally, we shall show that there exists an open neighborhood \(M'\) of 0 in \(M''\) such that \(F_t: X_t \to Y\) is locally stable for any \(t \in M'\). The mapping \(\omega F: \Theta_{Y \times M} \to F^* \Theta_{Y \times M}\) (cf.§2) induces naturally a mapping \(\pi_Y^* \Theta_Y \to \Theta_{Y}^p\), where \(\pi_Y\) denotes the projection: \(Y \times M \to Y\). We define a sheaf homomorphism \(\overline{\omega F}: \pi_Y^* \Theta_Y \to F_{\mathcal{G}} \Theta\) to be the composition of the mappings \(\pi_Y^* \Theta_Y \to F_\mathcal{G} \Theta_{Y}^p\) and \(F_{\mathcal{G}} \Theta_{Y}^p \to F_{\mathcal{G}} \Theta\). We set \(\mathcal{H} := F_{\mathcal{G}} \Theta_{Y}^p / \overline{\omega F}(\pi_Y^* \Theta_Y)\). Then \(\mathcal{H}\) is a coherent sheaf on \(Y \times M''\): hence \(\text{Supp} \mathcal{H}\) is an analytic subvariety of \(Y \times M''\). We claim that \(\mathcal{H}_{(0,0)} = 0\) for any point \((q, 0) \in Y \times M''\). This is proved as follows:

We set \(S := F^{-1}(q, 0) \cap \Sigma = F^{-1}(q) \cap \Sigma(f)\), and let \(p_1, \ldots, p_s\) be distinct points of \(S\). Since \(F_0 = f\) is locally stable, \(F\) is equivalent to \(f \times \text{id}_M\) on an open neighborhood of \(S\) in \(\mathcal{G}\). Hence by Lemma (5.1) \(F\) is locally stable at \(S\), so locally infinitesimally stable at \(S\). Therefore, observing that

\[
\mathcal{H}_{(0,0)} \hookrightarrow \Theta_{Y,0} / tF(\Theta_{X,0}) + \omega F(\pi_Y^* \Theta_Y),
\]

where \(\Theta_{Y,0} = \Theta_{X,(p_1,0)} \times \cdots \times \Theta_{X,(p_s,0)}\) and \(\Theta_{X,0} = \Theta_{X,(p_1,0)} \times \cdots \times \Theta_{X,(p_s,0)}\), we conclude that \(\mathcal{H}_{(0,0)} = 0\) for any \((q, 0) \in Y \times M''\) (cf. Remark (5.1)). This means \((Y \times 0) \cap \text{Supp} \mathcal{H} = \emptyset\). Well, by the assumption that \(Y\) is compact, the projection \(\pi_{M''}: Y \times M'' \to M''\) is a proper mapping, so \(\pi_{M''}(\text{Supp} \mathcal{H})\) is an analytic subvariety of \(M''\). Since \((Y \times 0) \cap \text{Supp} \mathcal{H} = \emptyset\), one has \(0 \notin \pi_{M''}(\text{Supp} \mathcal{H})\). Hence there exists an open neighborhood \(M'\) of 0 in \(M''\) such that \(M' \cap \pi_{M''}(\text{Supp} \mathcal{H}) = \emptyset\), so \((Y \times M'') \cap \text{Supp} \mathcal{H} = \emptyset\). This means \(\mathcal{H}_{(0,0)} = 0\) for any \((q, t) \in Y \times M'\); equivalently,

\[
tF(\Theta_{X,0}) + \omega F(\pi_Y^* \Theta_Y) = \Theta_{X,0}
\]

for any \((q, t) \in Y \times M'\) where \(S_q = F^{-1}(q, t) \cap \Sigma\), which is a finite set of \(X_t\). From this it follows that if \(t \in M'\) one has

\[
tF(\Theta_{X,0}) + \omega F(\Theta_Y) = F_t^* \Theta_{Y,0}
\]

for any \(q \in Y\) and any finite subset of \(S\) of \(F^{-1}(q)\), because the map \((tF_p)_*: \Theta_{X,p} \to F_t^* \Theta_{Y,p}\) is a submersion at any point \(p\) in \(X_t \Sigma\). Therefore we conclude that \(F_t\) is locally infinitesimally stable, equivalently, locally stable for any \(t \in M'\).

Q. E. D.

Chapter II: Analytic subvarieties with ordinary singularities

§6. Definition of analytic subvarieties with ordinary singularities and the classification of ordinary singularities

1°) Let \(Y\) be a complex manifold, and \(Z\) a proper analytic subvariety of \(Y\) of a pure dimension. Let \(\nu: X \to Z\) be the normalization of \(Z\). We set \(f = \nu \circ \chi: X \to Y\), where \(\chi\) denotes the inclusion map \(Z \to Y\).

Definition (6.1): In the above situation we say \(Z\) is an analytic subvariety with ordinary singularities of \(Y\) if the following are satisfied:

(i) \(X\) is non-singular;
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(ii) \( f = \epsilon \circ v \): \( X \to Y \) is a locally stable holomorphic map.

**Definition (6.2):** Let \( Y \) be a complex manifold. A germ \((Y, Z, q)\) of an analytic subvariety of \( Y \) is an equivalence class of analytic subvarieties \( Z \) in an open neighborhood \( D \) of \( q \) in \( Y \) with \( q \in Z \), where \( Z \subset D \subset Y \) and \( Z' \subset D' \subset Y \) are equivalent if there is an open neighborhood \( D'' \subset D \cap D' \) of \( q \) in \( Y \) such that \( Z \cap D'' = Z' \cap D'' \). If \( Z \) is a member of an equivalence class \((Y, Z, q)\) we call \( Z \) a representative of \((Y, Z, q)\), and say \((Y, Z, q)\) is the germ of \( Z \) at \( q \).

If there is no fair of confusion, we shall mix up the concepts of a germ and a representative of it frequently.

**Definition (6.3):** Let \( Y \) be an \( m \)-dimensional complex manifold, and \( Z \) an \( n \)-dimensional analytic subvariety with ordinary singularities of \( Y \). For any point \( q \) in \( Z \) we call a germ \((Y, Z, q)\) an \( n \)-dimensional ordinary singularity in an \( m \)-dimensional ambient manifold.

**Definition (6.4):** Let two germs \((Y, Z, q)\) and \((Y', Z', q')\) of analytic subvarieties be given, we say \((Y, Z, q)\) and \((Y', Z', q')\) are isomorphic if there exists a germ \( h: (Y, q) \to (Y', q') \) of an invertible holomorphic map with \( h(q) = q' \), such that \((Y', h(Z), q')\) coincides with \((Y', Z', q')\) as a germ of an analytic subvariety.

We consider the problem to classify all of ordinary singularities up to isomorphisms. The purpose of this section is to show that it is equivalent to classify all of simultaneously stable multi-germs of holomorphic maps (cf. Definition (1.5)), with the dimensions of source manifolds \( < \) those of target manifolds, up to isomorphisms. An isomorphism between two multi-germs of holomorphic maps is defined naturally as follows:

**Definition (6.5):** Let two multi-germs \( f: (X, S) \to (Y, T) \) and \( g: (X', S') \to (Y', T') \) of holomorphic maps be given. We say \( f \) and \( g \) are isomorphic if there exist multi-germs \( h: (X, S) \to (X', S') \) and \( k: (Y, T) \to (Y', T') \) of invertible holomorphic maps such that \( g = k \circ f \circ h^{-1} \).

2) Let a germ \((Y, Z, q)\) of an ordinary singularity be given. We mix up the concepts of a germ and a representative of it, so we may consider \( Z \) is an analytic subvariety with ordinary singularities in the complex manifold \( Y \). Let \( v: X \to Z \) be the normalization of \( Z \). We set \( f = \epsilon \circ v: X \to Y \), where \( \epsilon \) denotes the inclusion map \( Z \to Y \), and \( S := f^{-1}(q) \). Then by the definition of an analytic subvariety with ordinary singularities, the multi-germ \( f: (X, S) \to (Y, q) \) of a holomorphic map is simultaneously stable.

**Definition (6.6):** For a germ \((Y, Z, q)\) of an "ordinary singularity" we call the above simultaneously stable multigerm \( f: (X, S) \to (Y, q) \) of a holomorphic map the normalization of \((Y, Z, q)\).

We note that by the uniqueness of a normalization up to isomorphisms, the normalization \( f: (X, S) \to (Y, q) \) of \((Y, Z, q)\) is uniquely determined up to isomorphisms. More generally we have the following.

**Proposition (6.1):** Let \((Y, Z, q)\) and \((Y', Z', q')\) be germs of analytic subvarieties with ordinary singularities. Let \( f: (X, S) \to (Y, q) \) (resp. \( g: (X', S') \to (Y', q') \)) be the normalization of \((Y, Z, q)\) (resp. \((Y', Z', q')\)). Then \((Y, Z, q)\) and \((Y', Z', q')\) are isomorphic if, and only if, \( f: (X, S) \to (Y, q) \) and \( g: (X', S') \to (Y', q') \) are isomorphic.
Proof: It is obvious that if $f: (X, S) \rightarrow (Y, q)$ and $g: (X', S') \rightarrow (Y', q')$ are isomorphic, then $(Y, Z, q)$ and $(Y', Z', q')$ are isomorphic. Conversely, we assume that $(Y, Z, q)$ and $(Y', Z', q')$ are isomorphic. We denote by $\psi: (Y, q) \rightarrow (Y', q')$ a germ of an invertible holomorphic map which gives an isomorphism between $(Y, Z, q)$ and $(Y', Z', q')$. $\psi$ induces a germ of an analytic isomorphism $\psi_{1Z}: (Z, q) \rightarrow (Z', q')$. We denote by $\nu: X \rightarrow Z$ (resp. $\nu': X' \rightarrow Z'$) the normalization of $Z$ (resp. the normalization of $Z'$). We can lift the analytic isomorphism $\psi_{1Z}: (Z, q) \rightarrow (Z', q')$ to the normalizations ([4], Proposition (2.28)), so there is a germ of an analytic isomorphism $\phi: (X, S) \rightarrow (X', S')$ such that $\nu = \psi_{1Z} \circ \phi^{-1}$. That is, $f: (X, S) \rightarrow (Y, q)$ and $g: (X', S') \rightarrow (Y', q')$ are isomorphic.

Q. E. D.

Proposition (6.1) means that there is an injective map from the set of equivalence classes of germs of ordinary singularities into the set of equivalence classes of simultaneously stable multi-germs of holomorphic maps with the dimensions of source manifolds < those of target manifolds. To show that this map is surjective we need the following proposition.

Proposition (6.2): Let $f: (X, S) \rightarrow (Y, q)$ be a simultaneously stable multi-germ of a holomorphic map with $\dim X < \dim Y$. Then there exist open neighborhoods $U$ of $S$ in $X$ and $V$ of $q$ in $Y$ with $f(U) \subseteq V$, such that the following hold:

(i) $f_{1U}: U \rightarrow V$ is a finite map, i.e., proper and $f_{1U}(q')$ is a finite set in $V$ for any point $q'$ in $f_{1U}(U)$;

(ii) $f_{1U}: U \rightarrow V$ is a locally stable holomorphic map.

Proof: To show the assertion (i) it is enough to show that there exist such $U$ and $V$ at each point of $S$, so we assume that $S$ is one point, say $p$. We claim that $p$ is an isolated point of $f^{-1}(q)$. Suppose the contrary, the dimension of $f^{-1}(q)$ at $p$ is not less than one. Let $A$ be an irreducible component of $f^{-1}(q)$ which contains $p$, and whose dimension at $p$ is not less than one. We endow $A$ with the structure of a reduced complex space. We set $\mathcal{E} := \Theta_{X|A}$ and $\mathcal{E}' := f^*\Theta_{Y|A}$, and consider the map $tf_{1A}: \mathcal{E} \rightarrow \mathcal{E}'$, the restriction of the Jacobian map of $f$ to $A$. We set $\mathcal{F} := tf_{1A}(\mathcal{E})$. Well, since $f$ is stable at $p$, one has

$$tf(\Theta_{X,p}) + wf(\Theta_{Y,q}) = f^*\Theta_{Y,p}.$$ 

Note that for any local holomorphic vector field $\zeta = \sum_{j=1}^{m} \zeta_j(y) \left( \frac{\partial}{\partial y_j} \right)$ defined on an open neighborhood of $q$ in $Y$, the restriction of the pull-back $wf(\zeta) = \sum_{j=1}^{m} \zeta_j(f(x)) f^* \left( \frac{\partial}{\partial y_j} \right)$ of $\zeta$ to $A$ is a constant vector field. Hence, restricting the equality

$$tf(\Theta_{X,p}) + wf(\Theta_{Y,q}) = f^*\Theta_{Y,p}$$

to $A$, we have $\mathcal{F}_p + C^m = \mathcal{E}_{|p}$, where we identify $\mathcal{E}_{|p}$ with $\Theta_{X|p}$ ($m = \dim Y$) and regard $C^m$ ($\subseteq \Theta_{X|p}$) as a submodule of $\mathcal{E}_{|p}$. From this it follows dim$_C \mathcal{E}_{|p}/\mathcal{F}_p \leq m$. Now we consider the following sequence of $C$-vector subspaces of $\mathcal{E}_{|p}/\mathcal{F}_p$:

$$\mathcal{E}_{|p}/\mathcal{F}_p \supset m_{p}^{0}(\mathcal{E}_{|p}/\mathcal{F}_p) \supset m_{p}^{1}(\mathcal{E}_{|p}/\mathcal{F}_p) \supset \cdots.$$ 

Since dim$_C \mathcal{E}_{|p}/\mathcal{F}_p < \infty$, there is a non-negative integer $k$ such that

$$m_{p}^{k}(\mathcal{E}_{|p}/\mathcal{F}_p) = m_{p}^{k+1}(\mathcal{E}_{|p}/\mathcal{F}_p) = \cdots.$$ 

Then by Nakayama's Lemma, we conclude that $m_{p}^{b}(\mathcal{E}_{|p}/\mathcal{F}_p) = 0$, which is equivalent to $m_{p}^{b} \mathcal{E}_{|p} \subset \mathcal{F}_p$. We denote by $\mathcal{I}(p)$ the ideal sheaf of $p$ in $\mathcal{E}_{|A}$; i.e., the sheaf of germs of
holomorphic functions on $A$ which vanish at $p$. Since the sheaf $\mathcal{S}(p)^k \mathcal{E}$ is coherent, there exist an open neighborhood $U$ of $p$ in $A$ and cross-sections $\phi_1, \ldots, \phi_t \in \Gamma(U, \mathcal{S}(p)^k \mathcal{E})$ such that $(\phi_1)_{p'}, \ldots, (\phi_t)_{p'}$ generate $(\mathcal{S}(p)^k \mathcal{E})_{p}$ for any point $p'$ in $U$. In view of the fact that $(\phi_1)_{p'}, \ldots, (\phi_t)_{p'} \in \mathcal{S}(p)^k \mathcal{E}$ and $\mathcal{F}_p \supset m_p^e \mathcal{E}$, we infer that there exists an open neighborhood $U'$ of $p$ in $A$ such that $\phi_{1|U'}, \ldots, \phi_{t|U'} \in \Gamma(U', \mathcal{F})$.

Then, since $(\phi_1)_{p'}, \ldots, (\phi_t)_{p'}$ generate $(\mathcal{S}(p)^k \mathcal{E})_{p'} = \mathcal{E}_{p'}$ for any point $p'$ in $U'$ with $p' \neq p$, we have $\mathcal{E}_{p'} \subset \mathcal{F}_{p'}$, so $\mathcal{E}_{p'} = \mathcal{F}_{p'}$ at any point $p'$ in $U'$ with $p' \neq p$. On the other hand $\mathcal{E}_{p'} \neq \mathcal{F}_{p'}$ at any point $p'$ in $A$. Indeed, $(\eta_{1|p'})_{p'}: \mathcal{E}_{p'} \to \mathcal{E}_{p'}$ is not surjective at any point $p'$ in $A$, since $\dim X < \dim Y$. Thus we have a contradiction. Consequently we conclude that there is no irreducible component of $f^{-1}(q)$ of dimension $\geq 1$ which contains $p$; equivalently $p$ is an isolated point of $f^{-1}(q)$. Therefore there are open neighborhoods $U$ of $p$ in $X$ and $V$ of $q$ in $Y$ with $f(U) \subset V$, such that $f_{|U}: U \to V$ is a finite map ([4], Lemma (3.2)). Next we will prove the assertion (ii). We set

$$\mathcal{E} := (f_{|U})^* \Theta_V / \iota(f_{|U})(\Theta_U),$$

which is a coherent sheaf on $U$. Since $f_{|U}: U \to V$ is a proper map, the direct image sheaf $(f_{|U})_* \mathcal{E}$ is also coherent on $V$. We denote by $\omega(f_{|U}): \Theta_V \to (f_{|U})_* \mathcal{E}$ the homomorphism of sheaves induced by the map $\omega(f_{|U}): \Theta_{V,q} \to ((f_{|U})_* \Theta_V)_{f^{-1}(q)}$ defined at any point $q'$ in $V$ (cf. Chapter 1, §2).

We set $\mathcal{H} := (f_{|U})_* \mathcal{E} / \omega(f_{|U})(\Theta_V)$, which is a coherent sheaf on $V$. Note that $f_{|U}: U \to V$ is a finite map, so if we take $U$ sufficiently small, we may assume that $S = U \cap f^{-1}(q)$. Then we have $\mathcal{H} = 0$, so $f_{|U}: U \to V$ is simultaneously stable at $S$, equivalently, simultaneously infinitesimally stable at $S$. Then there exists an open neighborhood $V'$ of $q$ in $V$ such that $\mathcal{H} = 0$ for any point $q'$ in $V'$, since the sheaf $\mathcal{H}$ is coherent. We set $U' = (f_{|U})^{-1}(V')$. The fact $\mathcal{H} = 0$ for any point $q'$ in $V'$ is equivalent to that

$$\iota(f_{|U})(\Theta_{U', S}) + \omega(f_{|U})(\Theta_{V', q}) = ((f_{|U})^* \Theta_{V', S})$$

for any point $q'$ in $V'$, where $S = f_{|U}(U')$. This means $f_{|U}: U' \to V'$ is locally infinitesimally stable, so locally stable. This completes the proof.

By Proposition (6.1) and Proposition (6.2) we obtain the following.

**Proposition (6.3):** There is a bijective correspondence between the set of equivalence classes of germs of ordinary singularities and the set of equivalence classes of simultaneously stable multi-germs of holomorphic maps with the dimensions of source manifolds < those of target manifolds.

**Proof:** All that remain is to prove the correspondence which assigns its normalization $f: (X, S) \to (Y, q)$ to each germ $(Y, Z, q)$ of an ordinary singularity.

Let a simultaneously stable multi-germ $f: (X, S) \to (Y, q)$ with $\dim X < \dim Y$ be given. Then by proposition (6.2) there exist open neighborhoods $U$ of $S$ in $X$ and $V$ of $q$ in $Y$ with $f(U) \subset V$, such that the map $f_{|U}: U \to V$ is a finite map, and locally stable. Especially, the map $f_{|U}: U \to V$ is proper, so by Remmert's proper mappings theorem, $Z := f_{|U}(U)$ is an analytic subvariety of $V$; and by definition it is an analytic subvariety with ordinary singularities of $V$. We denote by $X \to Z \to Y$ the natural factorization of $f: X \to Y$. Then by Corollary (4.2) $f': X \to Z$ is the normalization of $Z$. Therefore we are done.

Q. E. D.
3°) For any multi-germ \( f: (X, S) \to (Y, q) \) of a holomorphic map, we define \( C \)-algebras:

\[
R(f) := \mathcal{O}_{X,S}/f^*m_q \mathcal{O}_{X,S},
\]

where \( m_q \) denotes the maximal ideal in \( \mathcal{O}_{Y,q} \);

\[
R_k(f) := R(f)/\mathcal{M}^{k+1},
\]

where \( k \) is a non-negative integer and \( \mathcal{M} \) denotes the intersection of the maximal ideals in \( R(f) \) (which is a semi-local ring, i.e., has only finitely many maximal ideals); and

\[
\bar{R}(f) := \lim R_k(f)
\]

We note that the canonical decomposition of \( \mathcal{O}_{X,S} \) into a Cartesian product

\[
\mathcal{O}_{X,S} = \mathcal{O}_{X,p_1} \times \cdots \times \mathcal{O}_{X,p_s}
\]

(where \( p_1, \ldots, p_s \) are the distinct points of \( S \)) give rise to a canonical decomposition of \( R(f) \) and \( R_k(f) \) into a Cartesian product

\[
R(f) \simeq R(f_1) \times \cdots \times R(f_s), \text{ and }
\]

\[
R_k(f) \simeq R_k(f_1) \times \cdots \times R_k(f_s)
\]

respectively, where \( f_i = f_{l(x,p_i)}: (X, p_i) \to (Y, q) \) for \( 1 \leq i \leq s \).

Given another multi-germ \( f': (X', S') \to (Y', q') \) of a holomorphic map, we let \( p'_1, \ldots, p'_s \) be the distinct points of \( S' \). With these notations J. N. Mather showed the following:

**Theorem (6.1):** ([17], Theorem A). Suppose that \( f \) and \( f' \) are simultaneously stable, and that \( s = s' \), \( \dim X = \dim X' \), \( \dim Y = \dim Y' \). Then \( f \) is isomorphic to \( f' \) (cf. Definition (6.5)) if, and only if, there is a permutation \( \tau \) of \( \{1, \ldots, s\} \) and an isomorphism \( R_{m+1}(f_i) \simeq R_{m+1}(f'_{\tau(i)}) \) of \( C \)-algebras for \( 1 \leq i \leq s \) where \( m = \dim Y = \dim Y' \).

In addition to this theorem J. N. Mather gave a necessary and sufficient condition for any finite set \( A_1, \ldots, A_s \) of the \( C \)-algebras of the quotients of formal power series rings over \( C \), under which there exists a simultaneously stable multi-germ \( f: (X, S) \to (Y, q) \) such that \( A_i \simeq R(f_i) \) \( (1 \leq i \leq s) \), where \( S \) is a set with \( s \) points \( p_1, \ldots, p_s \) and \( f_i = f_{l(x,p_i)}: (X, p_i) \to (Y, q) \) for \( 1 \leq i \leq s \) ([17], Theorem B). In view of these J. N. Mather's theorems and Proposition (6.3) we may say that the classification of ordinary singularities has been already done by J. N. Mather in some sense.

§ 7. Examples of higher dimensional ordinary singularities

1°) In [20] J. N. Mather gave the complete classification of the \( C \)-algebras \( A \) of the quotients of formal power series rings over \( C \) for which there exist stable germs \( f: (X, p) \to (Y, q) \) of holomorphic maps such that \( A \simeq R(f) \), when the pair \( (\dim X, \dim Y) \) of positive integers belongs to the "nice range" (cf. Definition (3.3)) and satisfy the inequality \( \dim X < \dim Y \). By use of this J. N. Mather's classification and "Normal form Theorem for stable germs" ([17], Theorem (5.10)), we will give some examples of higher dimensional ordinary singularities.

**Proposition (7.1):** Let \( f: (X, S) \to (Y, q) \) be a multi-germ of a holomorphic map.
Let $p_1, \ldots, p_s$ be the distinct points of $S$. We set $f_i := f_{i(x,p_0)} : (X, p_i) \to (Y, q)$, i.e., the germ at $p_i$ of $f$, and

$$C(f_i) := \{ p' \in X \mid R_m(f)_{p'} \simeq R_m(f_i) \}$$

where $m = \dim Y$, and $R_m(f)_{p'}$ denotes the $C$-algebra $R_m(f)$ of the germ $f_{p'} := f_{i(x,p')}$ : $(X, p') \to (Y, f(p'))$ of a holomorphic map (cf. §6, 3°). Then $f$ is simultaneously stable at $S$ if, and only if, the following two conditions are satisfied:

(i) for every $i (1 \leq i \leq s)$ $f_i$ is stable at $p_i$;

(ii) $(df_i)_{p_i} (T_{p_i} C(f_i))$, $\ldots$, $(df_s)_{p_s} (T_{p_s} C(f_s))$ are in a general position in $T_Y Y$ (cf. Definition (3.4)), where $(df_i)_{p_i} : T_{p_i} X \to T_{p_i} Y$ denotes the Jacobian map of $f$ at $p_i$, and $T_{p_i} C(f_i)$ the tangent space at $p_i$ of $C(f_i)$ for $1 \leq i \leq s$.

(We note that if the condition (i) is satisfied, then by Theorem (3.2) $j^m f_i$ intersects $C_{s_i}$ (the contact class in $J^m(X, Y)$ containing $s_i := j^m f_i(p_i)$) transversely; hence $C(f_i) = (j^m f)^{-1}(C_{s_i})$ is a submanifold in a sufficiently small open neighborhood of $p_i$ in $X$.)

Proof: Let $U_1, \ldots, U_s$ be open neighborhoods of $p_1, \ldots, p_s$ in $X$, respectively, which are mutually disjoint. We set $s = (s_1, \ldots, s_s)$, and $C_{s_i} = C_{s_i} \cap J^m(U_i, Y)$ for $1 \leq i \leq s$, then the contact class $C_{s_i}$ containing $s_i$ in $J^m(U_1, Y) \times \cdots \times J^m(U_s, Y)$ is identical with $C_{s_1} \times \cdots \times C_{s_s} \cap \beta^{-1}(\Delta^{(s)} Y)$, where

$$
\Delta^{(s)} Y = \{ (q_1, \ldots, q_s) \in Y \times \cdots \times Y \ (s\text{-times}) \mid q_1 = \cdots = q_s \}; \quad \text{and} \quad \beta = \beta_1 \times \cdots \times \beta_s : J^m(U_1, Y) \times \cdots \times J^m(U_s, Y) \longrightarrow Y \times \cdots \times Y \ (s\text{-times})
$$

denotes the target map. Note that if $f$ is simultaneously stable at $S$, then by definition each $f_i$ is stable at $p_i$ for $1 \leq i \leq s$, and that by Theorem (3.2) this is equivalent to that $j^m f_i \cap C_{s_i}$ at $p_i$ for $1 \leq i \leq s$. So it is enough to show that, under the assumption that $j^m f_i \cap C_{s_i}$ at $p_i$ for $1 \leq i \leq s$, the condition (ii) is equivalent to that $f$ is simultaneously stable at $S$. Hence we assume that $j^m f_i \cap C_{s_i}$ at $p_i$ for $1 \leq i \leq s$. Then, applying Lemma (3.2) for $C_{s_i} = W_1, \ldots, C_{s_i} = W_s$ and $k_1 = \cdots = k_s = m$, we conclude that the condition (ii) above is equivalent to $j^m f_1 \times \cdots \times j^m f_s \cap C_{s_i}$ at $(p_1, \ldots, p_s)$. Well, this is equivalent to that $f$ is simultaneously stable at $S$ by Theorem (3.2).

Q.E.D.

2°) In this and next two paragraphs, we shall use a notation

$$C(A) := \{ p \in X \mid \hat{R}(p) \simeq A \}$$

for any holomorphic map $g : X \to Y$ between complex manifolds and a $C$-algebra $A$ of the quotient of formal power series ring over $C$, where $R(g)_p$ denotes the $C$-algebra $\hat{R}(g)_p$ of the germ $g_p = g_{i(x,p)} : (X, p) \to (Y, g(p))$ of a holomorphic map.

Let $f : (C^n, 0) \to (C^n, 0)$ be a germ of a holomorphic map. If we restrict ourselves to the case where the pair $(n, m)$ of positive integers belongs to the "nice range", and satisfies $n < m$ and $n \leq 2/3 \cdot m + 1$, then $f$ is simultaneously stable if, and only if, $R(f)$ is isomorphic to one of the following $C$-algebras:

$$A_0 := C[[x]]/(x^n), \quad A_1 := C[[x]]/(x^2) \quad \text{and} \quad A_2 := C[[x]]/(x^3), \quad $$

where $C[[x]]$ denotes the formal power series rings over $C$ in $x$ (cf. [20]). In order to calculate the defining equations of ordinary singularities, we need to give the "normal form" of $f$, an explicit description of $f$ in local coordinates, as to each case by use of "Normal form theorem for stable germs" in [17], and furthermore we need to give
the defining equations of the locus \( C(A_i) \) for \( 1 \leq i \leq 2 \).

In the following we denote by \((x_1, ..., x_n)\) (resp. \((y_1, ..., y_n)\)) a linear coordinate system on \( C^n \) (resp. \( C^n \)).

a) \( \hat{R}(f) \cong A_0 \): The "normal form" of \( f \) is given by

\[
\begin{align*}
y_{i}f &= x_i & (1 \leq i \leq n) \\
y_{n}f &= 0 & (n + 1 \leq i \leq m),
\end{align*}
\]

that is, \( f \) is an embedding; and \( C(A_0) = C^n \).

b) \( \hat{R}(f) \cong A_1 \): The "normal form" of \( f \) is given by

\[
\begin{align*}
y_{i}f &= x_i & (1 \leq i \leq n - 1) \\
y_{n}f &= x_n^2 \\
y_{n+i}f &= x_i x_n & (1 \leq i \leq m - n \leq n - 1);
\end{align*}
\]

and \( C(A_1): x_1 = x_2 = \cdots = x_{n-1} = x_n = 0. \)

Note that this case occurs only when \( m - n \leq n - 1 \), i.e., \( m \leq 2n - 1 \).

c) \( \hat{R}(f) \cong A_2 \): The "normal form" of \( f \) is given by

\[
\begin{align*}
y_{i}f &= x_i & (1 \leq i \leq n - 1) \\
y_{n}f &= x_n^3 + x_1 x_n \\
y_{n+i}f &= x_2 x_n + x_{2i+1} x_n^2 & (1 \leq i \leq m - n);
\end{align*}
\]

and \( C(A_2): x_1 = x_2 = \cdots = x_{2(m-n)+1} = x_n = 0. \) Note that this case occurs only when \( 2(m-n)+1 \leq n - 1 \), i.e., \( m \leq \frac{2}{3} \left( n - \frac{2}{3} \right) \).

3°) If the pair \((n, m)\) of positive integers satisfies the inequality \( m \geq 2n - 1 \), then it is included in the case we have treated in the previous paragraph. By the calculation there and Proposition (7.1), we have the following propositions, whose proof will be omitted. We refer to the proof of Proposition (7.7) in the next paragraph.

**Proposition (7.2):** If \( m \geq 2n + 1 \), then an \( n \)-dimensional analytic subvariety with ordinary singularities in an \( m \)-dimensional complex manifold is non-singular.

**Proposition (7.3):** An \( n \)-dimensional analytic subvariety with ordinary singularities in a \( 2n \)-dimensional complex manifold is locally isomorphic to one of the following:

1. non-singular point;
2. the union of two \( n \)-planes which intersect transversely at a point (ordinary double point).

**Proposition (7.4):** An \( n \)-dimensional analytic subvariety \((n \geq 2)\) with ordinary singularities in a \((2n-1)\)-dimensional complex manifold is locally isomorphic to one of the following:

1. non-singular point;
2. the union of two \( n \)-planes which intersect transversely along a line in \( C^{2n-1} \) (ordinary double point);
3. the germ of an analytic subvariety in \( C^{2n-1} \) at 0 defined by the equations:

\[
y_{n+i} y_{n+j} - y_i y_j y_n = 0 & (1 \leq i \leq j \leq n - 1),
\]

where \((y_1, ..., y_{2n-1})\) is a linear local coordinate system on \( C^{2n-1} \) (cuspidal point);
4. the union of three planes which intersect transversely at a point in \( C^3 \). (This
occurs only when \( n = 2 \).)

**Remark (7.1):** J. Roberts showed in [27] that if one projects an \( n \)-dimensional algebraic manifold embedded in a sufficiently higher dimensional projective space into \( 2n \)-dimensional (resp. \((2n-1)\)-dimensional projective space) by a "generic" linear projection, then the singularities of the image are as described in Proposition (7.3) (resp. in Proposition (7.4)).

4°) We will consider the case of hypersurfaces with ordinary singularities. The pairs \((n, m)\) of positive integers with \( m = n + 1 \) included in the cases treated in paragraph 2° of this section are: \((1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\). Note that the singularity of a locally stable holomorphic map such that \( \tilde{R}(f) \simeq A_2 = \mathbb{C}[[x]]/(x^3) \) occurs only when \((n, m) = (4, 5) \) or \((5, 6)\) among these cases. The cases \((n, m) = (1, 2), (2, 3)\) are included in Proposition (7.3) and Proposition (7.4), respectively. Concerning the remaining cases, we have the following propositions. We will give a proof only for Proposition (7.7). The other two propositions can be proved by the same arguments. In the following \((y_1, \ldots, y_n)\) denotes a linear coordinate system on \( \mathbb{C}^n \), and we make a convention that \( P(y) = 0 \) means the germ of an analytic subvariety at 0 in \( \mathbb{C}^n \) defined as the zero locus of a polynomial \( P(y) \) in \( y_1, \ldots, y_n \) with \( P(0) = 0 \).

**Proposition (7.5):** A 3-dimensional analytic subvariety with ordinary singularities in a 4-dimensional complex manifold is locally isomorphic to one of the following:

(1) non-singular point;
(1)_k the union of \( k \) 3-planes which intersect transversely at 0 in \( \mathbb{C}^4 \) (2 \( \leq k \leq 4 \)) (k-ple point);
(2) \( y_2^3 - y_3^2 y_4 \equiv 0 \) (cuspidal point);
(1) + (2) \( y_2(y_2^2 - y_3^2 y_4) = 0 \).

**Proposition (7.6):** A 4-dimensional analytic subvariety with ordinary singularities in a 5-dimensional complex manifold is locally isomorphic to one of the following:

(1) non-singular point;
(1)_k the union of \( k \) 4-planes which intersect transversely at 0 in \( \mathbb{C}^5 \) (2 \( \leq k \leq 5 \)) (k-ple point);
(2) \( y_3^3 - y_2^2 y_4 = 0 \) (cuspidal point);
(1) + (2) \( y_2(y_2^2 - y_3^2 y_4) = 0 \);
(1)_2 + (2) \( y_2y_3(y_2^2 - y_3^2 y_4) = 0 \);
(3) \( y_3^3 + 2y_1y_3y_3 + (y_1^2y_3^2 - 3y_2y_3y_4 + y_1y_3^2)y_3 - (y_3^2y_4 + y_2(y_2^2 + y_1y_3^2))y_4 = 0 \).

**Proposition (7.7):** A 5-dimensional analytic subvariety with ordinary singularities in a 6-dimensional complex manifold is locally isomorphic to one of the following:

(1) non-singular point;
(1)_k the union of \( k \) 5-planes which intersect transversely at 0 in \( \mathbb{C}^6 \) (2 \( \leq k \leq 6 \)) (k-ple point);
(2) \( y_2^3 - y_1 y_5 = 0 \) (cuspial point);
(1)+(2) \( y_2 (y_2^2 - y_1 y_5) = 0; \)
(1)+(2) \( y_2 y_5 (y_2^2 - y_1 y_5) = 0; \)
(1)+(2) \( y_2 y_5 y_4 (y_2^2 - y_1 y_5) = 0; \)
(2) \( (y_2^2 - y_1 y_5) (y_2^2 - y_1 y_5) = 0; \)
(3) \( y_2^3 + 2 y_1 y_3 y_5 + (y_1^2 - 3 y_2 y_3 y_5 + y_1 y_3) y_6 - (y_2^2 y_5 + y_2 (y_2 + y_1 y_3)) y_5 = 0; \)
(1)+(3) \( y_4 [y_2^3 + 2 y_1 y_3 y_5 + (y_1^2 - 3 y_2 y_3 y_5 + y_1 y_3) y_6 - (y_2^2 y_5 + y_2 (y_2 + y_1 y_3)) y_5] = 0. \)

**Proof:** Let \( Z \) be a 5-dimensional analytic subvariety with ordinary singularities in a 6-dimensional complex manifold \( Y \). Then by definition there exists a 5-dimensional complex manifold \( X \) and a locally stable holomorphic map \( f: X \to Y \) such that \( f(X) = Z \). By Corollary (4.1) \( f: X \to Y \) is a finite map, so for each point \( p \) in \( X \) there exist neighborhoods \( U \) of \( p \) in \( X \) and \( V \) of \( f(p) \) in \( Y \) such that \( f(U) = V \) is a proper map and \( f^{-1}(f(p)) \cap U = \{ p \} \). Taking such open neighborhoods \( U \) and \( V \) for each point \( p \) in \( X \), we calculate the local equation of \( f(U) \) in \( V \).

In the case we have been treating now the \( C \)-algebra \( R(f)_p \) is isomorphic to one of the following:

\[ A_1 = \mathbb{C}[[x]]/(x), \quad A_1 = \mathbb{C}[[x]]/(x^2), \quad A_2 = \mathbb{C}[[x]]/(x^3) \]

at any point \( p \) in \( X \) because of the arguments in the previous paragraph.

If \( p \in C(A_1) \), then \( f \) is an immersion at \( p \); so \( f(p) \) is a non-singular point of \( f(U) \) and the codimension of \( (df)_p (T_p C(A_0)) \) in \( T_{f(p)} Y \) is equal to 1.

If \( p \in C(A_2) \), then \( f \) is explicitly described as:

\[
\begin{align*}
y_2 f &= x_i \quad (1 \leq i \leq 4) \\
y_3 f &= x_i^2 \\
y_6 f &= x_1 x_5
\end{align*}
\]

in adequate local coordinate systems \( (x_1, \ldots, x_5) \) on \( U \) centered at \( p \) and \( (y_1, \ldots, y_6) \) on \( V \) centered at \( f(p) \); so \( f(U) \) is defined by \( y_2^2 - y_1 y_5 = 0 \) at \( f(p) \). Note that the locus \( C(A_1) \cap U \) is defined by \( x_1 = x_5 = 0; \) \( f(C(A_1)) \cap V \) is defined by \( y_1 = y_5 = y_6 = 0; \) and that the map \( f|_{C(A_1) U}: C(A_1) \cap U \to f(C(A_1)) \cap V \) is an isomorphism, so the codimension of \( (df)_p (T_p C(A_1)) \) in \( T_{f(p)} Y \) is equal to 3.

If \( p \in C(A_2) \), then \( f \) is explicitly described as:

\[
\begin{align*}
y_2 f &= x_i \quad (1 \leq i \leq 4) \\
y_3 f &= x_i^2 + x_i x_5 \\
y_6 f &= x_2 x_5 + x_3 x_3^2
\end{align*}
\]

where \( (x_1, \ldots, x_5) \) and \( (y_1, \ldots, y_6) \) are the same as above. The defining equation of \( f(U) \) at \( f(p) \) is calculated as follows:

Substituting \( x_1 = y_1, \ x_2 = y_2, \ x_3 = y_3 \) into the last two equations above, we have

\[
\begin{align*}
x_2^2 + y_1 x_5 - y_5 &= 0 \\
x_3 x_2^2 + y_2 x_5 - y_6 &= 0.
\end{align*}
\]

We regard this as a simultaneous equation for \( x_5 \) with coefficients in the polynomial ring \( C[y_1, \ y_2, \ y_3, \ y_5, \ y_6] \). We eliminate \( x_5 \) by calculating the resultant of the above equation (cf. [31]). Then we have
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\[ y_6^3 + 2y_1y_3y_6^2 + (y_1^3y_5^2 - 3y_2y_3y_5 + y_1y_2^2)y_6 - (y_3^2y_5 + y_2(y_1^2 + y_1y_2))y_6 = 0. \]

This is nothing but the local equation of \( f(U) \) at \( f(p) \) in \( V \). Note that the locus \( C(A_2) \cap U \) is defined by \( x_1 = x_2 = x_3 = x_5 = 0 \) at \( p \); \( f(C(A_2)) \cap V \) is defined by \( y_1 = y_2 = y_3 = y_5 = y_6 = 0 \); and that the map \( f_{|C(A_2) \cap U} : C(A_2) \cap U \rightarrow f(C(A_2)) \cap V \) is an isomorphism, so the codimension of \( (df)_p(T_pC(A_2)) \) in \( T_pY \) is equal to 5.

Now we will calculate the local equation of \( Z \) at each point in \( Z \). Let \( q \) be a point in \( Z \), and we set

\[ S := f^{-1}(q) = \{ p_1, \ldots, p_r, p_{r+1}, \ldots, p_{r+s}, p_{r+s+1}, \ldots, p_{r+s+t} \}. \]

Let \( U_1, \ldots, U_{r+s+t} \) be open neighborhoods of \( p_1, \ldots, p_{r+s+t} \) in \( X \), respectively, and \( V \) a neighborhood of \( f(p_i) = \cdots = f(p_{r+s+t}) = q \) in \( Y \) such that \( f_{|U_i} : U_i \rightarrow V \) is a finite map and \( f^{-1}(f(p_i)) \cap U_i = p_i \) for \( 1 \leq i \leq r+s+t \). Then one has \( Z \cap V = \bigcup_{i=1}^{r+s+t} f(U_i) \). The defining equation of each \( f(U_i) \) has already been calculated individually. We assume that \( p_1, \ldots, p_r \in C(A_0), \ p_{r+1}, \ldots, p_{r+s} \in C(A_1) \) and \( p_{r+s+1}, \ldots, p_{r+s+t} \in C(A_2) \). Since \( f \) is locally stable, it follows from Proposition (7.1) that

\[ 6 = \dim Y \geq \text{codim} \left\{ \bigcap_{i=1}^{r+s+t} (df)_{p_i}(T_{p_i}C(A_0)) \right\} \]

\[ = \sum_{i=1}^{r} \text{codim} (df)_{p_i}(T_{p_i}C(A_0)) + \sum_{i=r+1}^{r+s} (df)_{p_i}(T_{p_i}C(A_1)) + \sum_{i=r+s+1}^{r+s+t} (df)_{p_i}(T_{p_i}C(A_2)) \]

\[ = r + 3s + 5t. \]

The triples \((r, s, t)\) of non-negative integers satisfying the above inequality are:

1) \( 1 \leq r \leq 6, \ s = t = 0 \);
2) \( 0 \leq r \leq 3, \ s = 1, \ t = 0 \);
3) \( r = 0, \ s = 2, \ t = 0 \);
4) \( r = s = 0, \ t = 1 \);
5) \( r = 1, \ s = 0, \ t = 1 \).

If we determine the defining equation of \( Z \) in \( V \) so that \((df)_{p_1}(T_{p_1}C(A_0)), \ldots, (df)_{p_{r+s+t}}(T_{p_{r+s+t}}C(A_2))\) are in a general position in \( T_qY \) as to each case, then we have all of possible equations (1) + (1) + (3) of \( Z \) in the proposition.

Q. E. D.

§8. Families of locally trivial displacements of analytic subvarieties with ordinary singularities and their characteristic maps

1°) In this section we shall give the definition of an analytic family of locally trivial displacements of analytic subvarieties with ordinary singularities, and show that its characteristic map can be defined. These are the generalizations of the definitions of an analytic family of surfaces with ordinary singularities and its characteristic map given by K. Kodaira in [10]. Let \( Y \) be an \( m \)-dimensional compact complex manifold, and \( Z \) an \( n \)-dimensional irreducible analytic subvariety with ordinary singularities of \( Y \).

Definition (8.1): By a family of locally trivial displacements of \( Z \) in \( Y \) parameterized by an analytic variety \( M \), we means quintet \((Y \times M, Z', \pi, M, 0)\) where \( Y \times M \) is the product space of \( Y \) and \( M \), \( Z' \) is an analytic subvariety of codimension \((m-n)\) at
every point in $Y \times M$, $\pi$ is the restriction to $\mathcal{L}$ of the canonical projection $Pr_M : Y \times M \to M$, and $0$ is an assigned point of $M$, which satisfy the following conditions:

(i) for each point $t \in M$, the intersection $\mathcal{L} \cap (Y \times t) = \pi^{-1}(t)$ is an $n$-dimensional irreducible analytic subvariety with ordinary singularities of $Y$;

(ii) $\pi^{-1}(0)$ coincides with $Z$;

(iii) the canonical projection $Pr_M : Y \times M \to M$ is locally a projection of a product space also on $\mathcal{L}$; that is, for each point $p = (y, t) \in Y \times M$, there exists an open neighborhood $\mathcal{U}_p \subset Y \times M$ of $p$ and an isomorphism $\phi : \mathcal{U}_p \to U \times V$, where $U = \mathcal{U}_p \cap (Y \times t)$ and $V = Pr_M(\mathcal{U}_p)$, such that (a) the following diagram

\[
\begin{array}{ccc}
\mathcal{U}_p & \xrightarrow{\phi} & U \times V \\
\downarrow{Pr_M} & & \downarrow{Pr_V} \\
V & & V
\end{array}
\]

is commutative, (b) $\phi(\mathcal{U}_p \cap \mathcal{L}) = (U \cap \mathcal{L}) \times V$, and (c) $\phi_{|U \times t} = id_{U \times t}$.

2°) We denote by $\Theta_Y$ the sheaf of germs of holomorphic tangent vector fields on $Y$, and by $\Theta_Y(log Z)$ the sheaf of germs of logarithmic tangent vector fields along $Z$ on $Y$, i.e., the subsheaf of $\Theta_Y$ consisting of derivations of $\Theta_Y$ which send $\mathcal{S}(Z)$, the ideal sheaf of $Z$ in $\Theta_Y$, into itself.

Definition (8.2): We denote by $\mathcal{N}_{Z/Y}$ the quotient sheaf $\Theta_Y/\Theta_Y(log Z)$, and call this the sheaf of infinitesimal locally trivial displacements of $Z$ in $Y$.

Remark (8.1): Since $\mathcal{S}(Z)\Theta_Y \subset \Theta_Y(log Z)$, the sheaf $\mathcal{N}_{Z/Y}$ may be considered as a sheaf over $Z$; so $H^*(Y, \mathcal{N}_{Z/Y}) \simeq H^*(Z, \mathcal{N}_{Z/Y})$.

Remark (8.2): If $Z$ is a surface with ordinary singularities in a threefold $Y$, the sheaf $\mathcal{N}_{Z/Y}$ defined above coincides with the sheaf $\Psi$ defined by K. Kodaira in [10] as showed in [29], (I) (Proposition 1.2) there.

Restricting $M$ to an open neighborhood of $0$ in $M$ if necessary, we may assume the following:

(i) $M$ is an analytic subvariety of an open subset in $C^r$ with a system of local coordinates $t = (t_1, \ldots, t_r)$, and $0 = (0, \ldots, 0)$, the origin of $C^r$;

(ii) $Y$ is covered by a finite number of Stein coordinate neighborhoods $Y_i, i \in I$;

(iii) for each $Y_i$ there exist an open subset $\mathcal{V}_i$ of $Y_i \times M$ and an isomorphism $\phi_i : \mathcal{V}_i \to U_i \times M$, where $U_i = \mathcal{V}_i \cap (Y \times 0)$, such that (a) the diagram:

\[
\begin{array}{ccc}
\mathcal{V}_i & \xrightarrow{\phi_i} & U_i \times M \\
\downarrow{Pr_M|_{\mathcal{V}_i}} & & \downarrow{Pr_M} \\
M & & M
\end{array}
\]

is commutative, (b) $\phi_i(\mathcal{V}_i \cap \mathcal{L}) = (U_i \cap \mathcal{L}) \times M$, (c) $\phi_{i|U_i \times 0} = id_{U_i \times 0}$, (d) $U_i \cap \mathcal{L}$ is the zero locus of a finite number of holomorphic functions on $Y_i$;
(iv) \{\mathcal{U}_i\}_{i=0}^m is an open covering of \(Y \times M\).

Let \((y_1, \ldots, y_m)\) be a system of local coordinates on \(Y_i\). Since \(U_i \subset Y_i\), we may regard this as a system of local coordinates on \(U_i\). We denote by \((y_1, t)\) a set of \(m + r\) complex numbers \(y_1, \ldots, y_m, t, \ldots, t\), and also the point in \(Y_i \times C^r\) with the coordinates \((y_1, \ldots, y_m, t_1, \ldots, t_r)\). For each \(i\) we represent each isomorphism \(\phi_i: \mathcal{U}_i \to U_i \times M\) as \(\phi_i(y_1, t) = (Y_1(y_1, t), \ldots, Y_m(y_1, t), t)\) by using local coordinates \((y_1, t)\), where \(Y_1(y_1, t), \ldots, Y_m(y_1, t)\) are holomorphic functions on \(\mathcal{U}_i\). We have \(Y_i(y_1, 0) = y_1 (1 \leq \alpha \leq m)\) because of \(\phi_i(U_i \times 0) = i d_{U_i \times 0}\). We may consider \((Y_1(y_1, t), \ldots, Y_m(y_1, t), t)\) as another system of local coordinates on \(\mathcal{U}_i\). On the intersection \(\mathcal{U}_i \cap \mathcal{U}_j\) the coordinates functions \(Y_1, \ldots, Y_m\) are holomorphic functions of \(y_1, \ldots, Y_m, t\), so we write

\[(8.1)\ Y_i = F_{ij}(Y_1, \ldots, Y_m, t) (1 \leq \alpha \leq m).\]

We identify \(T_0(M)\), the tangent space of \(M\) at \(0\), with the subspace \(\{v \in T_0(C^r)| v(\mathcal{I}(M), 0) = 0\}\) of \(T_0(C^r)\), where \(\mathcal{I}(M), 0\) denotes the stalk at \(0\) of the ideal sheaf of \(M\) in \(\mathcal{O}_{C^r}\). For any \(\frac{\partial}{\partial t} \in T_0(M)\), we set

\[(8.2)\ \theta_i = \sum_{a=1}^{m} \frac{\partial Y_i}{\partial t} \bigg|_{t=0} \left( \frac{\partial}{\partial y_i^a} \right)\]

(where \(\frac{\partial Y_i}{\partial t} = \sum_{a=1}^{m} \theta_\lambda \frac{\partial Y_j}{\partial t} \underset{\lambda=0}{=} \sum_{a=1}^{m} \theta_\lambda \left( \frac{\partial}{\partial t} \right)\)),

which we regard as a holomorphic vector field on \(U_i = \mathcal{U}_i \cap (Y \times 0)\). Then we have

\[(8.3)\ \theta_i - \theta_j = \sum_{a=1}^{m} \left\{ \frac{\partial Y_i}{\partial t} \bigg|_{t=0} - \sum_{b=1}^{m} \frac{\partial Y_j}{\partial t} \bigg|_{t=0} \right\} \left( \frac{\partial}{\partial y_i^a} \right) \]

on \(U_i \cap U_j\). Well, by \((8.1)\) we have

\[(8.4)\ \frac{\partial Y_i}{\partial t} \bigg|_{t=0} = \sum_{a=1}^{m} \frac{\partial Y_i}{\partial t} \bigg|_{t=0} \frac{\partial F_{ij}}{\partial Y_j} \bigg|_{t=0} + \frac{\partial F_{ij}}{\partial y_i} \bigg|_{t=0}\]

Substituting \((8.4)\) into \((8.3)\), we have

\[(8.5)\ \theta_i - \theta_j = \sum_{a=1}^{m} \frac{\partial F_{ij}}{\partial y_i} \bigg|_{t=0} \left( \frac{\partial}{\partial y_i^a} \right) \]

on \(U_i \cap U_j\). Now we will prove that \(\theta_i - \theta_j \in \Gamma(U_i \cap U_j, \mathcal{O}_Y(\log Z))\). Let \(h_i(y_1) = \cdots = h_i(y_m) = 0\) be the defining equations of the anlytic subvariety \(U_i \cap \mathcal{X}\) in \(U_i\). Then the analytic subvariety \(\mathcal{U}_i \cap \mathcal{X}\) is defined by

\[h_i(Y(y_1, t)) = \cdots = h_i(Y(y_1, t)) = 0\]

because \(\phi_i(\mathcal{U}_i \cap \mathcal{X}) = (U_i \cap \mathcal{X}) \times M\). Hence for any \((y_1, t) \in \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{X}\), we have

\[h_i(F_{ij}(Y(y_1, t), t)) = \cdots = h_i(F_{ij}(Y(y_1, t), t)) = 0\]

We choose any point \(a_j = (a_1, \ldots, a_m) \in U_i \cap U_j \cap \mathcal{X}\) and fixed. Since \(\phi_j(\mathcal{U}_j \cap \mathcal{X}) = (U_j \cap \mathcal{X}) \times M\) is an isomorphism and \(\phi_j(\mathcal{U}_j \cap \mathcal{X}) = (U_j \cap \mathcal{X}) \times M\), there exists uniquely a point \((b_j(t), t) = (b_1(t), \ldots, b_m(t), t) \in \mathcal{U}_j \cap \mathcal{X}\) such that \(Y(b_j(t), t) = a_j, \ldots, Y_\alpha(b_j(t), t) = a_\alpha\) for any \(t \in M\). By implicit function theorem, each \(b_\alpha(t)\) is a holomorphic function in \(t\) with \(b_\alpha(0) = a_\alpha\) for \(1 \leq \alpha \leq m\). Therefore we have

\[h_i(F_{ij}(a_j, t)) = \cdots = h_i(F_{ij}(a_j, t)) = 0\]

Derivating these equations by \(t\) and substituting \(0\) for \(t\), we have
\[
\sum_{s=1}^{m} \frac{\partial F_{ij}}{\partial t}(a_j, 0) \cdot \frac{\partial h_{k_i}}{\partial y_i^s}(F_{ij}(a_j, 0)) = 0
\]
for \(1 \leq k \leq k_i\). We set
\[
(a_1^s, \ldots, a_r^s) = (F_{ij}(a_j, 0), \ldots, F_{ij}(a_j, 0)).
\]
Note that \((a_1^s, a_r^s)\) and \((a_1^s, \ldots, a_r^s)\) are the values of local coordinates functions \((y_1^s, \ldots, y_r^s)\) and \((y_1^s, \ldots, y_r^s)\), respectively, of the same point, say \(p\), in \(U_i \cap U_j\). Hence we have
\[
\sum_{s=1}^{m} \frac{\partial F_{ij}}{\partial t}(y_j(p), 0) \cdot \frac{\partial h_{k_i}}{\partial y_i^s}(y_j(p)) = 0
\]
for \(1 \leq k \leq k_i\). In view of (8.5) this is equivalent to \((\theta_i - \theta_j) h_i^s = 0\) on \(U_i \cap U_j \cap \mathcal{Z}\); so we conclude that \(\theta_i - \theta_j \in \Gamma(U_i \cap U_j, \Theta_T(\log Z))\). Therefore the collection \(\{Q(\theta_i)\}_{i=1}^n\) defines an element of \(H^0(Y, \mathcal{N}_{Z/Y})\), so an element of \(H^0(\mathcal{Z}, \mathcal{N}_{Z/Y})\) (cf. Remark (8.1)), where \(Q\) denotes the map \(\Gamma(U_i, \Theta_T) \rightarrow \Gamma(U_i, \mathcal{N}_{Z/Y})\) induced by the natural homomorphism of sheaves: \(\Theta_T \rightarrow \mathcal{N}_{Z/Y}\).

**Definition (8.3):** We define the characteristic map \(\sigma: T_0(M) \rightarrow H^0(\mathcal{Z}, \mathcal{N}_{Z/Y})\) of a family \((Y \times M, \mathcal{Z}, \pi, M, 0)\) of locally trivial displacements of \(Z\) in \(Y\) parametrized by an analytic variety \(M\) by \(\sigma \left( \left( \frac{\partial}{\partial t} \right) \right) = \{Q(\theta_i)\}_{i=1}^n \) for any \(\left( \frac{\partial}{\partial t} \right) \in T_0(M)\).

**Proposition (8.1):** The characteristic map \(\sigma\) defined above is independent of the choice of coverings and systems of coordinates.

**Proof:** Obviously \(\sigma\) is invariant under a refinement of coverings. Hence it suffices to consider the case of fixed coverings.

Let \((y_i')\) be another system of coordinates on \(Y_i\), and let \(\hat{\phi}(y'_i) = (\hat{Y}_i(y'_i, t) \ldots, \hat{Y}_r(y'_i, t), t)\) be another isomorphism from \(U_i\) onto \(U_i \times M\) with \(\hat{\phi}|_{U_i \times 0} = id_{U_i \times 0}\). On the intersection \(U_i \cap U_j\), \(Y_1, \ldots, Y_r\) are holomorphic functions of \(\hat{Y}_1, \ldots, \hat{Y}_r\), \(t\), so we write

\[
(8.6) \quad Y_i = G_i(\hat{Y}_i, t) \quad (1 \leq i \leq m) .
\]

We set \(\partial_t = \sum_{s=1}^{m} \frac{\partial Y_i^s}{\partial t} \bigg|_{t=0} \left( \frac{\partial}{\partial y_i^s} \right)\). From (8.6) it follows that

\[
\frac{\partial Y_i^s}{\partial t} \bigg|_{t=0} = \sum_{s=1}^{m} \frac{\partial G_i^s}{\partial \hat{Y}_i^s} \bigg|_{t=0} + \frac{\partial G_i^s}{\partial t} \bigg|_{t=0}
\]

Hence we have

\[
\theta_i = \sum_{s=1}^{m} \frac{\partial Y_i^s}{\partial t} \bigg|_{t=0} \left( \frac{\partial}{\partial y_i^s} \right) \quad (\text{cf. } (8.2))
\]

\[
= \sum_{s=1}^{m} \frac{\partial G_i^s}{\partial \hat{Y}_i^s} \bigg|_{t=0} \left( \sum_{s=1}^{m} \frac{\partial G_i^s}{\partial Y_i^s} \bigg|_{t=0} \frac{\partial}{\partial y_i^s} \right) + \sum_{s=1}^{m} \frac{\partial G_i^s}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial y_i^s}
\]

By the equality (8.6), we have

\[
y_i = G_i(\hat{y}_i', 0), \quad (1 \leq i \leq m) .
\]

From this it follows that

\[
\frac{\partial}{\partial y_i^s} = \sum_{s=1}^{m} \frac{\partial y_i^s}{\partial y_i^s} \frac{\partial}{\partial y_i^s} = \sum_{s=1}^{m} \frac{\partial G_i^s}{\partial \hat{y}_i^s} \bigg|_{t=0} \frac{\partial}{\partial y_i^s}
\]

Substituting this equality into (8.7), we have
\[ \theta_i = \sum_{\beta=1}^{m} \frac{\partial Y_\beta}{\partial t} \bigg|_{t=0} - \sum_{a=1}^{n} \frac{\partial G_i^a}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial y_i^a} \]

\[ = \theta_i + \sum_{a=1}^{n} \frac{\partial G_i^a}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial y_i^a} \]

Well, by the same arguments which was used to prove that \( \theta_i - \theta_j = \sum_{a=1}^{n} \frac{\partial F_i^a}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial y_i^a} \)

is an element of \( \Gamma(U_1 \cap U_j, \Theta_4(\log Z)) \), we can prove that \( \sum_{a=1}^{n} \frac{\partial G_i^a}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial y_i^a} \)

is an element of \( \Gamma(U_i, \Theta_Y(\log Z)) \). Therefore we conclude that \( Q(\theta_i) = Q(\theta_j) \) in \( \Gamma(U_i, \mathcal{N}_{Z/Y}) \), and so the collections \( \{Q(\theta_i)\}_{i=1}^{m} \) and \( \{Q(\theta_j)\}_{j=1}^{m} \) defines the same element of \( H^0(Z, \mathcal{N}_{Z/Y}) \).

Q.E.D.

Remark (8.3): Observing the proof till now, we note that for the purpose to define the characteristic map it suffices for a given family \( \mathcal{Z} \) to be "locally trivial", and that \( \mathcal{Z} \) does not need to consist of analytic subvarieties with 'ordinary singularities'.

Chapter III: Relation between deformations of locally stable holomorphic maps and locally trivial displacements of analytic subvarieties with ordinary singularities

§9. Relation between characteristic maps

1°) Definition (9.1): For any holomorphic map \( f: X \rightarrow Y \) we denote by \( \mathcal{F}_{X/Y} \) the cokernel of the canonical homomorphism \( tf: \Theta_X \rightarrow f_*\Theta_Y \) of sheaves (cf. Chapter I, §2), and call this the sheaf of infinitesimal deformations of \( f: X \rightarrow Y \) (cf. [8]).

Proposition (9.1): Let \( f: X \rightarrow Y \) be a proper locally stable holomorphic map between complex manifolds with \( \dim X < \dim Y \). We set \( Z = f(X) \), i.e., an irreducible analytic subvariety with ordinary singularities of \( Y \). Then there exists canonically an isomorphism \( \mathcal{N}_{Z/Y} \rightarrow f_*\mathcal{F}_{X/Y} \) of sheaves, where \( \mathcal{N}_{Z/Y} \) is the sheaf of infinitesimal locally trivial displacements of \( Z \) in \( Y \) (cf. Definition (8.2)).

Proof: By the definition of the sheaf \( \mathcal{N}_{Z/Y} \) there exists an exact sequence of sheaves:

\[ 0 \rightarrow \Theta_Y(\log Z) \rightarrow \Theta_Y \rightarrow \mathcal{N}_{Z/Y} \rightarrow 0, \]

where \( \Theta_Y(\log Z) \) denotes the sheaf of germs of logarithmic tangent vector fields along \( Z \) (cf. §8, 2°). On the other hand, since \( f: X \rightarrow Y \) is a locally stable holomorphic map with \( \dim X < \dim Y \), by Corollary (4.1) it is a finite map; especially a non-degenerate holomorphic map. Hence the homomorphism \( tf: \Theta_X \rightarrow f_*\Theta_Y \) is injective, so the following sequence of sheaves is exact:

\[ 0 \rightarrow \Theta_X \rightarrow f_*\Theta_Y \rightarrow \mathcal{F}_{X/Y} \rightarrow 0. \]

Taking direct image of this sequence, we have an exact sequence of sheaves:

\[ 0 \rightarrow f_*\Theta_X \rightarrow f_*f_*\Theta_Y \rightarrow f_*\mathcal{F}_{X/Y} \rightarrow 0, \]

because \( f \) is a finite map. Comparing (9.1) and (9.2), we observe that there exists canonically a homomorphism \( \Theta_Y \rightarrow f_*f_*\Theta_Y \) of sheaves defined by

\[ \sum_{i=1}^{m} a_i(y) \left( \frac{\partial}{\partial y_i} \right) \rightarrow \sum_{i=1}^{m} a_i(f(x)) f^* \left( \frac{\partial}{\partial y_i} \right) \]
for any local cross-section \( \sum_{i=1}^m a_i(y) \left( \frac{\partial}{\partial y_i} \right) \) of the sheaf \( \Theta_Y \). This homomorphism is the same one as \( \omega f : \Theta_Y \to f_*\Theta_Y \) defined in Chapter I, §2 at each point \( q \in Z \). Hence we denote the above homomorphism \( \Theta_Y \to f_*f^*\Theta_Y \) by the same letter \( \omega f \). By the exact sequences of sheaves (9.1) and (9.2) we may think of \( \Theta_Y(log Z) \) and \( f_*\Theta_X \) as subsheaves of \( \Theta_Y \) and \( f_*f^*\Theta_Y \) respectively. We claim that the homomorphism \( \omega f : \Theta_Y \to f_*f^*\Theta_Y \) induces a homomorphism \( \Theta_Y(log Z) \to f_*\Theta_X \).

Indeed, we take a point \( q \in Y \), an element \( \vartheta_q \in \Theta_Y(log Z)_q \) and a local cross-section \( \theta \) of \( \Theta_Y(log Z)_q \), defined in a local coordinate neighborhood \( V \) of \( q \) in \( Y \), which represents \( \vartheta_q \) at \( q \). We set \( f^{-1}(q) = \{ p_1, \ldots, p_s \} \) (possibly empty). If we take \( V \) sufficiently small, then we may assume that \( f^{-1}(V) = \bigoplus_{i=1}^s U_i \) (disjoint sum), where each \( U_i \) is a local coordinate neighborhood of \( p_i \) in \( X \) for \( 1 \leq i \leq s \). Let \( f^{(i)} = f_{U_i} : U_i \to V \) and \( (x_1^{(i)}, \ldots, x_n^{(i)}) \) (resp. \( (y_1, \ldots, y_m) \)) a system of local coordinates on \( U_i \) (resp. \( V \)) centered at \( p_i \) (resp. \( q \)) for \( 1 \leq i \leq s \). We write \( \theta = \sum_{\beta=1}^m a_\beta(y) \left( \frac{\partial}{\partial y_\beta} \right) \). Then the image of \( \theta \) by the map \( \omega f : \)

\[
\Gamma(V, \Theta_Y(log Z)) \longrightarrow \Gamma(V, f_*f^*\Theta_Y) \\
\longrightarrow \Gamma(f^{-1}(V), f^*\Theta_Y) \\
\longrightarrow \bigoplus_{i=1}^s \Gamma(U_i, f^*\Theta_Y)
\]

induced by the homomorphism \( \omega f : \Theta_Y(log Z) \to f_*f^*\Theta_Y \) of sheaves is given by

\[
\omega f(\theta) = \left( \sum_{\alpha=1}^n a_\alpha(f^{(1)}(x))f^* \left( \frac{\partial}{\partial y_\alpha} \right), \ldots, \sum_{\beta=1}^m a_\beta(f^{(s)}(x))f^* \left( \frac{\partial}{\partial y_\beta} \right) \right).
\]

For our purpose it suffices to show that there exists \( \xi^{(i)} = \sum_{\alpha=1}^n a_\alpha(x^{(1)}(x))f^* \left( \frac{\partial}{\partial y_\alpha} \right) \in \Gamma(U_i, \Theta_X) \) such that \( tf(\xi^{(i)}) = \sum_{\alpha=1}^n a_\alpha(f^{(1)}(x))f^* \left( \frac{\partial}{\partial y_\alpha} \right) \) for \( 1 \leq i \leq s \), where \( tf \) denotes the map \( \Gamma(U_i, \Theta_X) \to \Gamma(U_i, f^*\Theta_Y) \) induced by the homomorphism \( tf : \Theta_X \to f^*\Theta_Y \) of sheaves. Since \( f \) is an immersion on \( X \setminus S(f) \), where \( S(f) = \{ p \in X | \text{Ker}(df)_p \neq 0 \} \), there exists \( \xi^{(i)} \in \Gamma(U_i, S(f), \Theta_X) \) for \( 1 \leq i \leq s \) such that \( tf(\xi^{(i)}) = \omega f(\vartheta) \) on \( U_i \setminus S(f) \). Here we recall that codim \( S(f) \geq 2 \), for \( f : X \to Y \) is a locally stable holomorphic map with \( \dim X < \dim Y \) (Corollary (4.2)). Hence by Riemann's Removable Singularity Theorem, there exists a unique holomorphic vector field \( \xi^{(i)} \) on \( U_i \) such that \( \xi^{(i)}|_{U_i \setminus S(f)} = \xi^{(i)} \) for \( 1 \leq i \leq s \). Furthermore, by Identity Theorem, \( tf(\xi^{(i)}) = \omega f(\vartheta) \) on \( U_i \) for each \( i \). This shows that the homomorphism \( \omega f : \Theta_Y \to f_*f^*\Theta_Y \) induces a homomorphism \( \Theta_Y(log Z) \to f_*\Theta_X \) as we have claimed above.

By this claim, comparing the sequences (9.1) and (9.2), we infer that \( \omega f : \Theta_Y \to f_*f^*\Theta_Y \) induces a homomorphism \( \mathcal{N}_{Z/Y} \to f_*\mathcal{F}_{X/Y} \) of sheaves, which we denote by \( \bar{f} \). Well, since \( f \) is locally stable, one has

\[
\bar{f}(\Theta_{X,f^{-1}(q)}) + \omega f(\Theta_{Y,q}) = f^*\Theta_{X,f^{-1}(q)}
\]

for any point \( q \) in \( Z \). This shows that the homomorphism \( \bar{f} : \mathcal{N}_{Z/Y} \to f_*\mathcal{F}_{X/Y} \) of sheaves defined above is surjective. Furthermore, \( \bar{f} \) is injective; so an isomorphism. Indeed, let \( \{ \vartheta \} \) be any element of \( \mathcal{N}_{Z/Y,q} \) for \( q \in Z \); let \( \vartheta_q \in \Theta_{Y,q} \) be such one whose image by the homomorphism \( \Theta_{Y,q} \to \mathcal{N}_{Z/Y,q} \) is \( \{ \vartheta \} \), and let \( \theta \) be a cross-section of \( \Theta_Y \) defined in
a local coordinate neighborhood $V$ of $q$ in $Y$ which represents $\theta_q$ at $q$. Let $p_1, \ldots, p_s, U_1, \ldots, U_s$, $f^{(i)} = f_{ij}: U_i \to V$ $(1 \leq i \leq s)$ be the same as before. We assume that $[\theta] = \operatorname{Ker} f$, then if we take $V$ sufficiently small, there exists
\[
\xi = (\xi^{(1)}, \ldots, \xi^{(s)}) \in \Gamma(f^{-1}(V), \Theta_Y) = \bigoplus_{i=1}^s \Gamma(U_i, \Theta_X)
\]
such that $tf(\xi) = \omega f(\theta)$ on $f^{-1}(V) = \bigoplus_{i=1}^s U_i$. To show that $f$ is injective, it suffices to show that $\theta \in \Gamma(V, \Theta_Y(\log Z))$ under the above condition; that is, it suffices to show that if $Z$ is defined by $h_1(x) = \cdots = h_s(x) = 0$ in $V$, where $h_1, \ldots, h_s$ are holomorphic functions on $V$, then $\theta h_i \equiv 0$ on $V \cap Z$ for $1 \leq i \leq t$. We represent $\theta$ and $\xi^{(i)}$ as $\theta = \sum_{\beta=1}^m a_\beta(y) \frac{\partial}{\partial y_\beta}$ and $\xi^{(i)} = \sum_{a=1}^n b_i^{(a)}(x) \frac{\partial}{\partial x_a}$ for $1 \leq i \leq s$, where $(y_1, \ldots, y_m)$ and $(x_1^{(1)}, \ldots, x_n^{(1)})$ $(1 \leq i \leq s)$ denote the same as before. Then $\omega f(\theta) = tf(\xi)$ means that
\[
a_\beta(f^{(i)}(x)) = \sum_{a=1}^n b_i^{(a)}(x) \frac{\partial f^{(i)}(x)}{\partial y_\beta} \tag{9.3}
\]
on $U_i$ for $1 \leq \beta \leq m$ and $1 \leq i \leq s$, where $f^{(i)} = y_\beta \circ f^{(i)}$. On the other hand, one has
\[
\theta h_i = \sum_{\beta=1}^m a_\beta(y) \frac{\partial h_i}{\partial y_\beta} \tag{9.4}
\]
on $V$ for each $\gamma$; hence by (9.3),
\[
(\partial h_i)(f^{(i)}(x)) = \sum_{\beta=1}^m \sum_{a=1}^n b_i^{(a)}(x) \frac{\partial f^{(i)}(x)}{\partial y_\beta} \frac{\partial h_i}{\partial x_a}(x) \tag{9.4}
\]
on $U_i$ for $1 \leq \gamma \leq t$ and $1 \leq i \leq s$. Well, since $h_i(f^{(i)}(y)) \equiv 0$ on $U_i$, differentiating this equation by $x_{a_i}^{(i)}$, one has
\[
\frac{\partial h_i}{\partial x_{a_i}^{(i)}}(f^{(i)}(x)) = \sum_{\beta=1}^m \frac{\partial f^{(i)}(x)}{\partial y_\beta} \frac{\partial h_i}{\partial x_{a_i}^{(i)}}(f^{(i)}(x)) \equiv 0 \tag{9.5}
\]
on $U_i$ for $1 \leq \gamma \leq t$ and $1 \leq i \leq s$. Then it follows from (9.4) and (9.5) that $(\partial h_i)(f^{(i)}(x)) \equiv 0$ on $U_i$ for every $\gamma$ and $i$, so $\theta h_i \equiv 0$ on $V \cap Z$ for $1 \leq \gamma \leq t$, because $V \cap Z = \bigcup_{i=1}^s f(U_i)$. Hence $\theta$ is surely an element of $\Gamma(V, \Theta_Y(\log Z))$.

Q.E.D.

2° Proposition (9.2): Let $f: X \to Y$ be a locally stable holomorphic map between compact complex manifolds with $\dim X < \dim Y$. We set $Z = \{f(X), \text{i.e.}, an\ irreducible \ analytic \ family \ of \ deformations \ of \ f: X \to Y \ parametrized \ by \ a \ analytic \ variety \ } M \text{ such that } \pi: Z \to M$ is a proper map (cf. Definition (5.1)). Then there exists an open neighborhood $M'$ of 0 in $M$ such that $(Y \times M', Z', \pi, M', 0)$ is an analytic family of locally trivial displacements of $Z$ in $Y$ parameterized by $M'$ (cf. Definition (8.1)), where $Z' = f(Z) \cap (Y \times M')$ and $\pi$ is the restriction to $Z'$ of the canonical projection $\text{Pr}_{M'}: Y \times M' \to M'$.

Proof: By Theorem (5.1) there exists an open neighborhood $M'$ of 0 in $M$ such that $F_t: X_t \to M$ is a locally stable holomorphic map for any $t \in M'$. Then by definition, $Z_t = F_t(X_t) \subset Y$ is an irreducible analytic subvariety with ordinary singularities of $Y$ for any $t \in M'$. Furthermore, by the definition of a locally stable holomorphic map, for any point $(p, t) \in Z^r$ there exist an open neighborhood $\mathcal{U}$ of $(p, t)$ in $Y \times M'$, an isomorphism $\phi: \mathcal{U} \to U \times V$ over $V$ (i.e. $\text{Pr}_{M'}|\mathcal{U} = \text{Pr}_{M'} \circ \phi$) with $\phi_{y_{\mathcal{U}}} = id_U$, where $U = \mathcal{U} \cap (Y \times t)$ and $V = \text{Pr}_{M'}(\mathcal{U})$, and an isomorphism $\psi: F^{-1}(\mathcal{U}) \to \tilde{U} \times V$ with $\psi_{\mathcal{U}} = id_{\mathcal{U}}$ where $\tilde{U} = \tilde{U}$.
$F^{-1}(\mathcal{Z}) \cap X_t$, such that the diagram

$$
\begin{array}{ccc}
F^{-1}(\mathcal{Z}) & \xrightarrow{\psi} & \tilde{U} \times V \\
F \downarrow & & \downarrow F_t \times \text{id}_V \\
\mathcal{Z} & \xrightarrow{\phi} & U \times V
\end{array}
$$

is commutative. From this it follows that the family $(Y \times M', \mathcal{Z}, \varpi, M', 0)$ is locally trivial.

Q.E.D.

Let $(\mathcal{Z}, F, \pi, M, 0)$ be an analytic family of deformations of a locally stable holomorphic map $f: X \to Y$ between compact complex manifolds with $\dim X < \dim Y$, such that (1) $\pi: \mathcal{Z} \to M$ is a proper map, and (2) $F_t: X_t \to Y$ is locally stable for any $t \in M$. We set

$$
\mathcal{Z} = F(\mathcal{Z}) \subset Y \times M,
$$

then by Proposition (9.2), $(Y \times M, \mathcal{Z}, \varpi, M, 0)$ is an analytic family of locally trivial displacement of $Z := f(X)$ in $Y$. Let $\tau_0: T_0(M) \to H^0(X, \mathcal{S}_{X/Y})$ be the characteristic map of the family $(\mathcal{Z}, F, \pi, M, 0)$ (cf. [8]), and let $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ be that of the family $(Y \times M, \mathcal{Z}, \varpi, M, 0)$ (cf. Definition (8.3)). We denote by the same letter $\mathfrak{f}$ the isomorphism $H^0(Z, \mathcal{N}_{Z/Y}) \cong H^0(X, \mathcal{S}_{X/Y})$ of cohomology groups induced by the isomorphism $\mathfrak{f}: \mathcal{N}_{Z/Y} \to \mathcal{S}_{X/Y}$ of sheaves in Proposition (9.1). (Note that the isomorphism $\mathfrak{f}: \mathcal{N}_{Z/Y} \to \mathcal{S}_{X/Y}$ of sheaves induces isomorphisms $H^*(Z, \mathcal{N}_{Z/Y}) \cong H^*(X, \mathcal{S}_{X/Y})$ of cohomology groups, since $f$ is a finite map.) Then the characteristic maps $\tau_0, \sigma_0$ are related through $\mathfrak{f}$ as follows:

Proposition (9.3): In the above situation, we have

$$
-\tau_0 = \mathfrak{f} \circ \sigma_0.
$$

Proof: For the family $(Y \times M, \mathcal{Z}, \pi, M, 0)$ above, restricting $M$ to a sufficiently small open neighborhood of 0 in $M$, we can take the open coverings $\{\mathcal{U}_i\}_{\mathcal{I}}$ of $Y \times M$,

$$
\{Y_i\}_{\mathcal{I}} \text{ and } \{U_i\}_{\mathcal{I}} \text{ of } Y,
$$

that was used for the proof of the existence $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ in §8, (2). We adopt the same notations as there. Let $h^1(y) = \cdots = h^k(y) = 0$ be the defining equations of $Z$ in $U_i$. We represent

$$
\phi_i: \mathcal{U}_i \longrightarrow U_i \times V
$$

by using local coordinates on $\mathcal{U}_i$ and $U_i \times M$, where $Y_i(y_i, t), \ldots, Y_k(y_i, t)$ are holomorphic functions on $\mathcal{U}_i$. Then the defining equations of $\mathcal{Z}$ in $\mathcal{U}_i$ are given by $h^1(y_i(t)) = \cdots = h^k(y_i(t)) = 0$, because $\phi_i(\mathcal{U}_i \cap \mathcal{Z}) = (U_i \cap \mathcal{Z}) \times M$.

Recall that the characteristic map $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ is defined by

$$
(9.6) \quad \sigma_0 \left( \frac{\partial}{\partial t} \right) = \{Q \left( \sum_{j=1}^n \frac{\partial Y_j}{\partial t} (y_i, 0) \left( \frac{\partial}{\partial y_j} \right) \right) \}_{i=1}^n
$$

for $\frac{\partial}{\partial t} \in T_0(M)$ with respect to the stein covering $\{U_i\}_{\mathcal{I}}$ of $Y$, where $Q$ denotes the map $\Gamma(U_i, \Theta_Y) \to \Gamma(U_i, \mathcal{N}_{Z/Y})$ induced by the natural homomorphism $\Theta_Y \to \mathcal{N}_{Z/Y}$ of sheaves.

We may assume that $\mathcal{Z}$ is covered by a finite number of stein coordinate neighborhoods $\mathcal{V}_j (j \in J)$, such that:

(i) there exists a map $A: J \to I$ such that $F(\mathcal{V}_j) \subset \mathcal{U}_{A(j)}$ for any $j \in J$;

(ii) for each $j \in J$ there exists an isomorphism $\psi_j: \mathcal{V}_j \to V_j \times M$,

where $V_j$ is a connected component of $X \cap f^{-1}(U_{A(j)})$;

(iii) for each $j \in J$, above $V_j$ is a stein coordinate neighborhood on $X$. 

```
We denote by \((x_j) = (x_j^1, \ldots, x_j^m)\) a system of local coordinates on \(V_j\). We may consider \((x_j, t)\) is a system of local coordinates on \(\mathcal{V}_j\). Let the map \(F\) be given by

\[
\begin{aligned}
    y_i^\beta &= F_i^\beta(x_j, t) \\
    t &= t
\end{aligned}
\]

on \(\mathcal{V}_j\) with respect to local coordinates on \(\mathcal{V}_j\) and \(\mathcal{B}_i\), where \(i = A(j)\). Then the characteristic map \(\tau_0: T_0(M) \to H^0(X, \mathcal{S}_{\mathcal{X}_{jY}})\) for the family \((\mathcal{X}, F, \pi, M, 0)\) is defined by

\[
\tau_0 \left( \frac{\partial}{\partial t} \right) = \left\{ P \left( \sum_{j=1}^n \frac{\partial F_i^\beta}{\partial t} (x_j, 0) f^* \left( \frac{\partial}{\partial y_i^\beta} \right) \right) \right\}_{i \in J}
\]

for \(\frac{\partial}{\partial t} \in T_0(M)\) with respect to the stein covering \(\{V_j\}_{j \in I}\) of \(X\), where \(P\) denotes the map \(\Gamma(V_j, f^* \mathcal{O}_Y) \to \Gamma(V_j, \mathcal{S}_{\mathcal{X}_{jY}})\) induced by the natural homomorphism \(f^* \mathcal{O}_Y \to \mathcal{S}_{\mathcal{X}_{jY}}\) of sheaves. By (9.6) and the definition of the map \(\mathfrak{f}: H^0(Z, \mathcal{N}_{\mathcal{X}}') \to H^0(X, \mathcal{S}_{\mathcal{X}_{jY}})\) (cf. Proposition (9.1)), we have

\[
(\mathfrak{f} \circ \sigma_0) \left( \frac{\partial}{\partial t} \right) = \left\{ P \left( \sum_{j=1}^n \frac{\partial Y_i^j}{\partial t} (f_j(x_j), 0) f^* \left( \frac{\partial}{\partial y_i^j} \right) \right) \right\}_{j \in J}
\]

with respect to the stein covering \(\{V_j\}_{j \in I}\) of \(X\), where \(f_j\) denotes the map \(f_j: V_j \to U_i\); so by this and (9.7), we have

\[
(9.8) \quad (\tau_0 + \mathfrak{f} \circ \sigma_0) \left( \frac{\partial}{\partial t} \right) = (P(\theta_j))_{j \in J},
\]

where we set

\[
\theta_j := \sum_{i=1}^n \left\{ \frac{\partial F_i^\beta}{\partial t} (x_j, 0) + \frac{\partial Y_i^j}{\partial t} (f_j(x_j), 0) \right\} f^* \left( \frac{\partial}{\partial y_i^\beta} \right)
\]

\(\in \Gamma(V_j, f^* \mathcal{O}_Y)\) for each \(j\). Well, since \(f\) is locally stable, taking open covering \(\{U_i\}_{i \in I}\) and \(\{V_j\}_{j \in J}\) sufficiently fine, we may assume that for each \(j\) there exist \(\xi_j \in \Gamma(V_j, \mathcal{O}_X)\) and \(\xi_j \in \Gamma(U_i, \mathcal{O}_X)\) such that

\[
(9.9) \quad tf(\xi_j) + \omega(\xi_j) = \theta_j.
\]

If we write \(\xi_j\) and \(\xi_j\) as \(\xi_j = \sum_{j=1}^n \xi_j^\beta(x_j) \left( \frac{\partial}{\partial x_j^\beta} \right)\) and \(\xi_j = \sum_{j=1}^n \xi_j^\beta(y_j) \left( \frac{\partial}{\partial y_j^\beta} \right)\), then (9.9) is equivalent to that

\[
(9.10) \quad \sum_{j=1}^n \xi_j^\beta(x_j) \left( \frac{\partial f_j^\beta}{\partial y_j^\beta} \right) (f_j(x_j)) + \xi_j^\beta(f_j(x_j)) = \frac{\partial F_i^\beta}{\partial t} (x_j, 0) + \frac{\partial Y_i^j}{\partial t} (f_j(x_j), 0)
\]

for \(1 \leq \alpha \leq m\), where \(f_j^\beta = y_j^\beta f_j\). From these equations we wish to derive that \(\xi \in \Gamma(U_i, \mathcal{O}_X(\log Z))\).

First, since \(F_j(x_j, t) = (F_j^1(x_j, t), \ldots, F_j^n(x_j, t))\) satisfies the equations

\[
h_j(Y(F_j(x_j, t), t)) = \cdots = h_j(Y(F_j(x_j, t), t)) = 0
\]

, differentiating these equations by \(t\) and substituting 0 for \(t\), we have

\[
(9.11) \quad \sum_{j=1}^n \left\{ \sum_{k=1}^n \frac{\partial F_k^\beta}{\partial t} (x_j, 0) + \frac{\partial Y_k^j}{\partial t} (f_j(x_j), 0) \right\} \cdot \frac{\partial h_j}{\partial y_j^\beta} (f_j(x_j)) = 0
\]

for \(1 \leq k \leq n\). Now, recall that \(Y(y_i, 0) = y_i\), because \(\phi_i |_{U_i} = id_{U_i}\); hence one has

\[
\frac{\partial y_i^\beta}{\partial y_j^\beta} (y_i, 0) = \delta_{i,j} \delta_{\alpha \beta}
\]

for \(1 \leq \alpha, \beta \leq m\). Therefore by (9.11), we have

\[
(9.12) \quad \sum_{j=1}^n \left\{ \frac{\partial F_j^\beta}{\partial t} (x_j, 0) + \frac{\partial Y_j^j}{\partial t} (f_j(x_j), 0) \right\} \cdot \frac{\partial h_j}{\partial y_j^\beta} (f_j(x_j)) = 0
\]
for $1 \leq k \leq k_i$.

Second, since $f_j(x) = (f_j'(x), \ldots, f_j''(x))$ satisfies the equations
\[ h^j(f_j(x)) = \cdots = h^j_1(f_j(x)) = 0, \]

differentiating these equations by $\xi_j$, we have
\[ \sum_{a=1}^{m} \xi_j^a(x) \left( \sum_{a=1}^{m} \left( \frac{\partial f_j^a}{\partial x_i} \right)(x) \left( \frac{\partial h^a}{\partial y_i} \right)(f_j(x)) \right) = 0 \]
for $1 \leq k \leq k_i$. From (9.10), (9.12) and (9.13) we derive that
\[ (\zeta_i h^j_0)(f_j(x)) = \sum_{a=1}^{m} \xi_j^a(f_j(x)) \left( \frac{\partial h^a}{\partial y_i} \right)(f_j(x)) = 0 \]
for $1 \leq k \leq k_i$. This holds for any $j \in J$ such that $V_j$ is a connected component of $X \cap f^{-1}(U_i)$. Hence we infer that $\zeta_i h^j = 0$ on $U_i \cap Z$ for $1 \leq k \leq k_i$; that is, $\zeta_i \in \Gamma(U_i, \Theta_X(\text{log } Z))$. Then, by the same arguments used to show that the homomorphism $\omega f: \Theta_f \to f^*f_* \Theta_f$ induces a homomorphism $\Theta_f(\text{log } Z) \to f^*f_* \Theta_f$ in the proof of Proposition (9.1), there exists $\xi_j \in \Gamma(V_j, \Theta_X)$ such that $f(\xi_j) = \omega f(\zeta_i)$. Hence by (9.9), one has $\theta_j = tf(\xi_j + \zeta_i)$. This means $P(\theta_j) = 0$ for each $j$. Therefore by (9.8) we conclude $	au_0 + f \circ \tau_0 = 0$. Q.E.D.

Remark (9.1): Proposition (9.1) and Proposition (9.3) are the generalizations of the results which E. Horikawa proved for surfaces in [9].

§10. Relative normalization theorem

In this section we shall consider the problem whether the converse of Proposition (9.2) in the previous section is true. Namely;

If a family $(Y \times M, \mathcal{Z}, \pi, M, 0)$ of locally trivial displacements of an irreducible analytic subvariety $Z$ with ordinary singularities of a compact complex manifold $Y$ parameterized by an analytic variety $M$ is given, then does there exist a family $(\mathcal{Z}, F, \pi', M, 0)$ of deformations of the locally stable holomorphic map $f: X \to Y$ such that $(Y \times M, F(\mathcal{Z}), w, M, 0) = (Y \times M, \mathcal{Z}, \pi, M, 0)$?

Here $f: X \to Y$ denotes the composite map of the normalization $\nu: X \to Z$ of $Z$ and the inclusion map $\iota: Z \subset Y$, and $w$ the restriction of the canonical projection $Pr_M: Y \times M \to M$ to $F(\mathcal{Z})$.

In the subsequence we shall show that the answer to this problem is "Yes".

Definition (10.1): Let $V$ be an analytic variety and we denote by $S(V) \subset V$ the singular locus. $h$ is called a weakly holomorphic function on $V$, if
(a) $h$ is a holomorphic function defined on $V \setminus S(V)$; and
(b) $h$ is locally bounded on $V$.

Definition (10.2): A simple analytic map $f: V_1 \to V$ between two analytic varieties $V_1$ and $V$ is a finite map, i.e., proper and $f^{-1}(q)$ is a finite set for any $q \in V$, such that there exist analytic subvarieties $A_1 \subset V_1$ and $A \subset V$ for which $V_1 - A_1$ and $V - A$ are dense in $V_1$ and $V$ respectively and the restriction $f: V_1 - A_1 \to V - A$ is a biholomorphic map.

Remark (10.1): Note that for a simple analytic map $f: V_1 \to V$ the image $f(V_1)$ is necessarily all of $V$; because $f(V_1)$ contains the dense open subset $V - A \subset V$ and $f(V_1)$ must be a closed subset of $V$, since the map $f$ is proper.
Lemma (10.1): Let \( f : V_1 \to V \) be a simple analytic map between analytic varieties. If \( V \) is normal, then \( f \) is a biholomorphic map.

Proof: In order to prove this it suffices to show that \( f \) is a homeomorphism and a canonical injective homomorphism \( \mathcal{O}_V \to f_* \mathcal{O}_{V_1} \) of sheaves is surjective. Since \( f \) is a simple analytic map, we may consider that the sheaf \( f_* \mathcal{O}_{V_1} \) is a subsheaf of the sheaf \( \tilde{\mathcal{O}}_V \) of germs of weakly holomorphic functions on \( V \). Well, since \( V \) is normal by assumption, we have \( \mathcal{O}_V = \tilde{\mathcal{O}}_V \). Therefore the canonical injective homomorphism \( \mathcal{O}_V \to f_* \mathcal{O}_{V_1} \) is surjective. Furthermore, from the fact \( \mathcal{O}_V \cong f_* \mathcal{O}_{V_1} \), it follows that \( f^{-1}(q) \) is connected, so one point for any \( q \in V_1 \). Hence \( f \) is injective, and so by Remark (10.1), \( f \) is bijective. Note that a proper bijective map is an open map. Therefore \( f \) is a homeomorphism between \( V_1 \) and \( V \). This completes the proof.

The answer to the problem explained at the beginning of this section follows from the following theorem.

Theorem (10.1): (Relative normalization theorem) Let \( \varpi : \mathcal{X} \to M \) be an analytic family of analytic varieties parametrized by an analytic variety \( M \), which is locally trivial at every point in \( \mathcal{X} \) in the following sense: for each point \( p \in \mathcal{X} \), there exists an open neighborhood \( \mathcal{U}_p \subset \mathcal{X} \) of \( p \) and an isomorphism \( \phi_p : \mathcal{U}_p \to U \times V \), where \( U = \mathcal{U}_p \cap Z_{\varpi(p)} \) \( (Z_{\varpi(p)} := \varpi^{-1}(\varpi(p)) \) and \( V = \varpi(\mathcal{U}_p) \), such that the following diagram

\[
\begin{array}{ccc}
\mathcal{U}_p & \xrightarrow{\phi_p} & U \times V \\
\downarrow{\varpi|_{\mathcal{U}_p}} & & \downarrow{\text{Pr}_V} \\
V & & V
\end{array}
\]

is commutative.

Then there exists an analytic family \( \pi : \mathcal{X} \to M \) of analytic varieties parametrized by the same analytic variety \( M \), and a surjective holomorphic map \( \varphi : \mathcal{X} \to \mathcal{X} \) over \( M \) (i.e., \( \pi = \varpi \circ \varphi \)) satisfying the following conditions:

(\( \alpha \)) \( \varphi_t : X_t \to Z_t \) is the normalization of \( Z_t \) for any \( t \in M \) \( (X_t := \pi^{-1}(t), \varphi_t := \varphi|_{X_t} : X_t \to Z_t) \);

(\( \beta \)) the map \( \varphi : \mathcal{X} \to \mathcal{X} \) is locally trivial in the following sense:

for any point \( p \in \mathcal{X} \), there exist an open neighborhood \( \mathcal{U}_p \) of \( p \) in \( \mathcal{X} \), an isomorphism \( \phi_p : \mathcal{U}_p \cong U \times V \) over \( V \), where \( U = \mathcal{U}_p \cap Z_{\varphi(p)} \) and \( V = \varphi(\mathcal{U}_p) \), and an isomorphism \( \psi_p : \varphi^{-1}(\mathcal{U}_p) \cong U^* \times V \) over \( V \), such that the diagram

\[
\begin{array}{ccc}
\varphi_p & \xrightarrow{\psi_p} & U^* \times V \\
\downarrow{\varphi^{-1}(\mathcal{U}_p)} & & \downarrow{\text{Id}_V \times \varphi^{-1}(\mathcal{U}_p)} \\
\mathcal{U}_p & \xrightarrow{\phi_p} & U \times V
\end{array}
\]

is commutative.

Furthermore, the above family \( \pi : \mathcal{X} \to M \) and the surjective holomorphic map \( \varphi : \mathcal{X} \to \mathcal{X} \) over \( M \) are uniquely determined up to isomorphisms over \( M \).

Proof: We set \( S(\varpi) := \{ p \in \mathcal{X} | \text{fiber } Z_{\varpi(p)} \text{ is not a manifold at } p \} \). Then since the
family \(\omega: \mathcal{Z} \to M\) is locally trivial, the set \(S(\omega)\) is an analytic subvariety of \(\mathcal{Z}\). We denote by \(\tilde{\partial}_{\mathcal{Z}}(\omega)\) the sheaf of germs of holomorphic functions on \(\mathcal{Z} \setminus S(\omega)\) which are locally bounded on \(\mathcal{Z}\). The proof is divided into three steps.

**Step 1**: First, we shall show that the sheaf \(\tilde{\partial}_{\mathcal{Z}}(\omega)\) is coherent. The problem is local in nature, so we may assume that the family \(\omega: \mathcal{Z} \to M\) is a product family \(\omega = Pr_M: \mathcal{Z} = Z \times M \to M\), where \(Z\) is a fiber of the family \(\omega: \mathcal{Z} \to M\). We denote by \(\nu_0: X \to Z\) the normalization of \(Z\), and set \(\mathcal{Z}' = X \times M, \ \pi' = Pr_M: \mathcal{Z}' \to M\) and \(v_0 = \nu_0 \times id_M: \mathcal{Z} \to \mathcal{Z}'\). In this situation we claim that \(v_0^* \tilde{\partial}_{\mathcal{Z}}(\omega) \cong \tilde{\partial}_{\mathcal{Z}'}(\omega)\) if we shrink \(\mathcal{Z}\) sufficiently small around a point in \(\mathcal{Z}\). If this is proved, since \(v\) is a proper mapping, \(\nu_0^* \tilde{\partial}_{\mathcal{Z}}\) is coherent; so \(\tilde{\partial}_{\mathcal{Z}}(\omega)\) is coherent as desired. The above claim is proved as follows:

Let \(a = (p_0, i_0)\) be any point in \(S(Z) \times M \subset \mathcal{Z}\), where \(S(Z)\) denotes the singular locus of \(Z\). Then there exists a holomorphic function \(d\), so called a universal denominator at \(a\), which is defined in an open neighborhood, say \(W_a\), of \(a\) in \(\mathcal{Z}\) and has the following properties:

1. if \(h\) is a weakly holomorphic function defined on an open neighborhood \(W'_a \subset W_a\) of \(a\), then \(d \cdot h\) has a unique holomorphic extension to \(W'_a\);
2. \(d\) does not vanish identically on any open subset of \(W_a\)

(cf. [26], Corollary 2 to Theorem 6).

We take such a universal denominator \(d\) defined on an open neighborhood \(W_a \subset \mathcal{Z}\) of \(a\). Then we have \(d_a \tilde{\partial}_{\mathcal{Z}}(\omega)_a = \sigma_{\mathcal{Z},a}\) (Note that one has \(\tilde{\partial}_{\mathcal{Z}}(\omega)_a = \tilde{\partial}(S(Z) \times M)_a = \tilde{\partial}_{\mathcal{Z},a}\) in the above situation, where \(\tilde{\partial}(S(Z) \times M)\) denotes the sheaf of germs of holomorphic functions on \(\mathcal{Z} \setminus S(Z) \times M\) which is locally bounded on \(\mathcal{Z}\), and \(\tilde{\partial}_{\mathcal{Z}}\) the sheaf of germs of weakly holomorphic functions on \(\mathcal{Z}\).) Therefore \(d_a \tilde{\partial}_{\mathcal{Z}}(\omega)_a\) is finitely generated as \(\sigma_{\mathcal{Z},a}\)-module, for \(\sigma_{\mathcal{Z},a}\) is a Noetherian ring. Well, by the property (2) above of the universal denominator \(d\), \(\tilde{\partial}_{\mathcal{Z}}(\omega)_a\) is isomorphic to \(d_a \tilde{\partial}_{\mathcal{Z}}(\omega)_a\) as \(\sigma_{\mathcal{Z},a}\)-module; hence \(\tilde{\partial}_{\mathcal{Z}}(\omega)_a\) is also finitely generated as \(\sigma_{\mathcal{Z},a}\)- module.

Let \((H_1)_a,\ldots, (H_m)_a \in \tilde{\partial}_{\mathcal{Z}}(\omega)_a\) be generators of \(\tilde{\partial}_{\mathcal{Z}}(\omega)_a\) as a \(\sigma_{\mathcal{Z},a}\)-module, and let \(H_1,\ldots, H_m\) be local cross-sections of \(\tilde{\partial}_{\mathcal{Z}}(\omega)\) which represent \((H_1)_a,\ldots, (H_m)_a\) respectively at \(a\). Shrinking \(W_a\) sufficiently small, we may assume that \(H_1,\ldots, H_m\) are defined on \(W_a\). We set \(W_a = W_a \setminus S(Z) \times M\). Then \(H_1,\ldots, H_m\) are holomorphic functions on \(W_a\). We define a holomorphic map \(H: \hat{W}_a \to \hat{W}_a \times C^m\) by

\[H(p, t) = ((p, t), H_1(p, t), \ldots, H_m(p, t))\]

for \((p, t) \in \hat{W}_a \subset Z \times M\). We set \(\hat{\omega}_a := H(\hat{W}_a)\) and \(\omega_a := \text{the closure of } \hat{\omega}_a \text{ in } W_a \times C^m\). We denote by \(\pi_a: \omega_a \to W_a\) the restriction of the canonical projection: \(W_a \times C^m \to W_a\) to \(\omega_a\). Note that since \(H_1,\ldots, H_m\) are bounded on \(W_a\), the map \(\pi_a\) is a finite map, i.e., proper and \(\pi_a^{-1}(t)\) is a finite subset of \(\omega_a\) for any \((p, t) \in W_a\). Furthermore, note that \(H: \hat{W}_a \to \hat{\omega}_a\) is a biholomorphic map; hence if we set \(\omega_a := \pi_a^{-1}(W_a)\), then one has \(\hat{\omega}_a = \omega_a | T_a\) and \(\pi_a|_{\omega_a}: \hat{\omega}_a \to \omega_a\) give an inverse map of the map \(H: \hat{W}_a \to \hat{\omega}_a\).

We claim that:

1. \((\pi_a)^{-1}(\hat{\omega}_a) \hookrightarrow \tilde{\partial}_{\mathcal{Z}}(\omega)_a\);
2. if we take \(W_a\) sufficiently small, then \(\pi_a: \omega_a \to W_a\) is isomorphic to \(v|_{\omega_a}: \mathcal{Z}_a := v^{-1}(W_a) \to W_a\) over \(W_a\), i.e., there exists an isomorphism \(\Phi: \omega_a \to \mathcal{Z}_a\) such that the diagram
is commutative.

Clearly there exists canonically a homomorphism \( v_\ast \mathcal{O}_X \to \hat{\mathcal{O}}_x(\mathcal{M}) \) of sheaves. The above claim assures that this homomorphism of sheaves gives an isomorphism \( (v_\ast \mathcal{O}_X)_a \cong \hat{\mathcal{O}}_x(\mathcal{M})_a \) at any point \( a \in \mathcal{X} \). From this fact it follows that the canonical homomorphism \( v_\ast \mathcal{O}_X \to \hat{\mathcal{O}}_x(\mathcal{M}) \) is surely an isomorphism. Thus, for the purpose to show that the sheaf \( \hat{\mathcal{O}}_x(\mathcal{M}) \) is coherent, it suffices to prove the above claim.

The proof of claim (1) above: Since \( \pi_a \downarrow \mathcal{X} : \mathcal{Y}_a \to \mathcal{W}_a \) is a biholomorphic map, there exists canonically an injective homomorphism \( ((\pi_a)_\ast \mathcal{O}_{\mathcal{Y}_a})_a \to \hat{\mathcal{O}}_x(\mathcal{M})_a \). Let \( h_a \) be any element of \( \hat{\mathcal{O}}_x(\mathcal{M})_a \), and \( \mathcal{H}_a \) a local cross-section of \( \hat{\mathcal{O}}_x(\mathcal{M}) \) defined in an open neighborhood of \( a \) in \( \mathcal{W}_a \), which represents \( h_a \) at \( a \). Since \( (H_1)_a, \ldots, (H_m)_a \) generate \( \hat{\mathcal{O}}_x(\mathcal{M})_a \) as \( \mathcal{O}_{\mathcal{X},a} \)-module, we can write \( h = \sum_{j=1}^m h_j \cdot H_j \) in an open neighborhood, say \( \mathcal{W}_a \) of \( a \) in \( \mathcal{W}_a \), where \( h_1, \ldots, h_m \) are holomorphic functions defined there. We set \( \tilde{h}(p, t, z) := \sum_{j=1}^m h_j(p, t) \cdot z_j \) for \( (p, t, z) \in \mathcal{W}_a \times \mathcal{C}^m \), where \( (z_1, \ldots, z_m) \) is a system of linear coordinates on \( \mathcal{C}^m \), then \( \tilde{h} \downarrow \pi_a^{-1}(\mathcal{W}_a) \) is a holomorphic function on \( \pi_a^{-1}(\mathcal{W}_a) \), and one has
\[
\tilde{h}(p, t, H_1(p, t) \ldots, H_m(p, t)) = \sum_{j=1}^m h_j(p, t) \cdot H_j(p, t)
\]
for any point \( (p, t) \) in \( \mathcal{W}_a \backslash \mathcal{S}(\mathcal{Z}) \times \mathcal{M} \); equivalently \( \tilde{h}(p, t, z) = h(p, a, p, t, z) \) for any point \( (p, t, z) \) in \( \pi_a^{-1}(\mathcal{W}_a) \backslash \mathcal{S}(\mathcal{Z}) \times \mathcal{M} \). The holomorphic function \( \tilde{h} \downarrow \pi_a^{-1}(\mathcal{W}_a) \) defines an element of \( (\pi_a_\ast \mathcal{O}_{\mathcal{Y}_a})_a \) which we denote by \( ((\pi_a)_\ast \tilde{h})_a \). Then the above fact shows that \( ((\pi_a)_\ast \tilde{h})_a \) is mapped to \( h_a \) by the canonical homomorphism \( ((\pi_a)_\ast \mathcal{O}_{\mathcal{Y}_a})_a \to \hat{\mathcal{O}}_x(\mathcal{M})_a \). This shows that the homomorphism \( ((\pi_a)_\ast \mathcal{O}_{\mathcal{Y}_a})_a \to \hat{\mathcal{O}}_x(\mathcal{M})_a \) is surjective, so one has \( ((\pi_a)_\ast \mathcal{O}_{\mathcal{Y}_a})_a \cong \hat{\mathcal{O}}_x(\mathcal{M})_a \).

The proof of claim (2) above: Since the sheaf \( \hat{\mathcal{O}}_Z \) of germs of weakly holomorphic functions on \( Z \) is coherent ([26], Theorem 4 in Chapter II), taking \( Z \) sufficiently small from the beginning, we may assume that there exist global cross-sections \( h_1, \ldots, h_n \) of the sheaf \( \hat{\mathcal{O}}_Z \) over \( Z \) such that \( (h_1)_p, \ldots, (h_n)_p \) generate \( \hat{\mathcal{O}}_{Z,p} \) as a \( \mathcal{O}_{\mathcal{X},p} \)-module at every point \( p \) in \( Z \). We set \( \tilde{Z} = \mathcal{Z} \cup \mathcal{S}(\mathcal{Z}) \), and define a map \( h : \tilde{Z} \to \mathcal{Z} \times \mathcal{C}^n \) by
\[
h(p) = (p, h_1(p), \ldots, h_n(p))
\]
for \( p \in \tilde{Z} \). Then the closure \( \overline{h(\tilde{Z})} \) of \( h(\tilde{Z}) \) in \( \mathcal{Z} \times \mathcal{C}^n \) is nothing but the normal model \( X \) of \( Z \), and the restriction of the natural projection \( \pi_\mathcal{Z} : \mathcal{Z} \times \mathcal{C}^n \to \mathcal{Z} \) to \( \overline{h(\tilde{Z})} \) coincides with the normalization \( \nu_0 : X \to \mathcal{Z} \) of \( Z \) ([26]). Well, since the germs of the cross-sections \( H_1, \ldots, H_n \in \Gamma(\mathcal{W}_a, \hat{\mathcal{O}}_x(\mathcal{M}))) \) taken before at the point \( a \) generate the stalk \( \hat{\mathcal{O}}_x(\mathcal{M})_a \) as a \( \mathcal{O}_{\mathcal{X},a} \)-module, regarding \( (h_1)_a, \ldots, h_n(a) \) as elements of \( \hat{\mathcal{O}}_x(\mathcal{M})_a \), we can write
(hi)_a = \sum_{j=1}^{m} (c_{ij})_a \cdot (H_j)_a

for 1 \leq i \leq n, where (c_{ij})_a \in \mathcal{O}_{x_a}. Hence, taking \mathcal{W}_a sufficiently small, we may assume that

h_i(p) = \sum_{j=1}^{m} c_{ij}(p, t) \cdot H_j(p, t)

holds for 1 \leq i \leq n on \mathcal{W}_a, where c_{ij}(p, t) are holomorphic functions there.

From now on we assume that \mathcal{W}_a is of the form \mathcal{U}_{p_0} \times \mathcal{V}_{t_0}, where \mathcal{U}_{p_0} \subset \mathbb{C} (resp. \mathcal{V}_{t_0} \subset M) is an open neighborhood of \mathcal{p}_0 (resp. \mathcal{t}_0) for \mathcal{a} = (\mathcal{p}_0, \mathcal{t}_0). We define a map \tilde{\Phi}: (\mathcal{U}_{p_0} \times \mathcal{V}_{t_0}) \times \Omega_m \to \mathcal{U}_{p_0} \times \mathcal{C}^m \times \mathcal{V}_{t_0} by

\tilde{\Phi}(\mathcal{p}, \mathcal{t}, z) = (\mathcal{p}, \sum_{j=1}^{m} c_{ij}(\mathcal{p}, \mathcal{t}) \cdot z_j, \ldots, \sum_{j=1}^{m} c_{ij}(\mathcal{p}, \mathcal{t}) \cdot z_j, \mathcal{t})

for (\mathcal{p}, \mathcal{t}, z) \in (\mathcal{U}_{p_0} \times \mathcal{V}_{t_0}) \times \Omega_m. By the definition of the map \tilde{\Phi} it is clear that

\tilde{\Phi}(\mathcal{Q}_a) = \nu_0^{-1}(\mathcal{U}_{p_0} \times \mathcal{V}_{t_0}) (\mathcal{Q}_a := \pi^{-1}(\mathcal{U}_{p_0} \times \mathcal{V}_{t_0})),

where we identify \mathcal{X} with the closure of \nu(\mathcal{Z}) in \mathcal{Z} \times \mathcal{C}^n and the normalization \nu_0: \mathcal{X} \to \mathcal{Z} with the restriction of the natural projection \pi_2: \mathcal{Z} \times \mathcal{C}^n \to \mathcal{Z}. We denote by \Phi the restriction \tilde{\Phi}|_{\mathcal{Q}_a}: \mathcal{Q}_a \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0}(\mathcal{U}_{p_0} := \nu_0^{-1}((\mathcal{U}_{p_0}))$, the normalization of \mathcal{U}_{p_0}) of the map \tilde{\Phi} to \mathcal{Q}_a.

With these notations we consider the following commutative diagram:

\[
\begin{array}{ccc}
\pi_a^{-1}(\mathcal{U}_{p_0} \times \mathcal{V}_{t_0}) = \mathcal{Q}_a & \overset{\Phi}{\longrightarrow} & \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} = \mathcal{X} = \mathcal{X} \times M \\
\pi_a \downarrow & & \downarrow \pi_a \downarrow \\
\pi_1 := \pi_1|_{\mathcal{U}_{p_0} \times \mathcal{V}_{t_0}} & & \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} \\
\nu_{1|\mathcal{U}_{p_0} \times \mathcal{V}_{t_0}} & \overset{\pi_a}{\longrightarrow} & \mathcal{V}_{t_0} \\
\end{array}
\]

We set \mathcal{U}_{p_0} = \mathcal{U}_{p_0}|_{S(\mathcal{Z})}, \mathcal{U}_{p_0} = \nu_0^{-1}(\mathcal{U}_{p_0}). Then the maps \pi_a|_{\tilde{\mathcal{Q}}_a}: \tilde{\mathcal{Q}}_a \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} and \nu_{1|\tilde{\mathcal{Q}}_a \times \mathcal{V}_{t_0}}: \tilde{\mathcal{Q}}_a \times \mathcal{V}_{t_0} \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} are biholomorphic maps; hence the map

\[
\Phi|_{\tilde{\mathcal{Q}}_a} = (\nu_{1|\tilde{\mathcal{Q}}_a \times \mathcal{V}_{t_0}})^{-1} \circ \pi_a|_{\tilde{\mathcal{Q}}_a}: \tilde{\mathcal{Q}}_a \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0}
\]

is so, too. Furthermore, since \pi_a: \mathcal{Q}_a \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} and \nu: \mathcal{Q}_a \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} are finite maps, it follows from the commutativity of the above diagram that the map \Phi: \mathcal{Q}_a \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} is also a finite map. Hence \Phi is a simple analytic map. We wish to show that \Phi is in fact a biholomorphic map.

We denote by \mu_0: \mathcal{V}_{t_0} \to \mathcal{V}_{t_0} the normalization of \mathcal{V}_{t_0}, and set

\[
\mu := id_{\mathcal{V}_{t_0}} \times \mu_0: \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0},
\]

\[
\mathcal{Q}_a = \mathcal{Q}_a \times \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} \mathcal{U}_{p_0} \times \mathcal{V}_{t_0} \mathcal{U}_{p_0} \times \mathcal{V}_{t_0}
\]

\[
\Phi' := Pr_{\mathcal{V}_{t_0} \times \mathcal{V}_{t_0}}: \mathcal{Q}_a \to \mathcal{U}_{p_0} \times \mathcal{V}_{t_0}.
\]
Deformations of Locally Stable Holomorphic Maps

\[ \mu' := \text{Pr}_{\mathcal{E}}: \mathcal{E}' \rightarrow \mathcal{E}, \]

\[ \pi'_1 := \text{Pr}_{V_{t_0}}: U_{t_0}^* \times V_{t_0}^* \rightarrow V_{t_0}^*, \text{ and} \]

\[ \pi'_2 := \pi'_1 \circ \Phi': \mathcal{E}' \rightarrow V_{t_0}^*. \]

Then we have the following commutative diagram:

\[ \begin{array}{ccc}
\mathcal{E} & \xrightarrow{\mu'} & \mathcal{E}' \\
\downarrow{\pi_2} & & \downarrow{\phi'} \\
U_{t_0}^* \times V_{t_0} & \xrightarrow{\mu} & U_{t_0}^* \times V_{t_0}^* \\
\downarrow{\pi_1} & & \downarrow{\pi'_2} \\
V_{t_0} & \xrightarrow{\mu_0} & V_{t_0}^*
\end{array} \]

Since \( \Phi \) and \( \mu \) are simple analytic maps, \( \Phi' \) is so, too. Then by Lemma (10.1), \( \Phi' \) is an isomorphism, for \( U_{t_0}^* \times V_{t_0}^* \) is normal. For each \( t \in V_{t_0} \) we denote by \( Y_{a,t} \) the fiber over \( t \in V_{t_0} \) of the analytic fiber space \( \pi_2: \mathcal{E} \rightarrow V_{t_0} \). Then we derive the following commutative diagram from the above one:

\[ \begin{array}{ccc}
Y_{a,t} & \xrightarrow{\mu'} & Y_{a,t} \times \mu_0^{-1}(t) & \xleftarrow{id_{Y_{a,t}} \times \iota} & Y_{a,t} \times (\mu_0^{-1}(t))_{\text{red}} \\
\downarrow{\phi} & & \downarrow{\pi'_1} & & \downarrow{\phi'} \\
U_{t_0}^* \times t & \xrightarrow{\mu} & U_{t_0}^* \times \mu_0^{-1}(t) & \xleftarrow{id_{U_{t_0}^*} \times \iota} & U_{t_0}^* \times (\mu_0^{-1}(t))_{\text{red}} \\
\downarrow{\pi_1} & & \downarrow{\pi'_2} & & \downarrow{\pi'_1} \\
V_{t_0} & \xrightarrow{\mu_0^{-1}(t)} & V_{t_0}^* & & (\mu_0^{-1}(t))_{\text{red}}
\end{array} \]

, where \((\mu_0^{-1}(t))_{\text{red}}\) denotes the underlying reduced complex space of \( \mu_0^{-1}(t) \), \( \iota: (\mu_0^{-1}(t))_{\text{red}} \rightarrow \mu_0^{-1}(t) \) the canonical injection, \( \Phi': Y_{a,t} \times (\mu_0^{-1}(t))_{\text{red}} \rightarrow U_{t_0}^* \times (\mu_0^{-1}(t))_{\text{red}} \) the pull-back of the map \( \Phi': Y_{a,t} \times \mu_0^{-1}(t) \rightarrow U_{t_0}^* \times \mu_0^{-1}(t) \) by the map \( \text{id}_{U_{t_0}^*} \times \iota: U_{t_0}^* \times (\mu_0^{-1}(t))_{\text{red}} \rightarrow U_{t_0}^* \times \mu_0^{-1}(t) \), and \( \pi'_1, \pi'_2 \) the projections to \((\mu_0^{-1}(t))_{\text{red}}\). Since \( \Phi': \mathcal{E} \rightarrow U_{t_0}^* \times V_{t_0}^* \) is an isomorphism over \( V_{t_0}^* \), the restriction \( \Phi': Y_{a,t} \times \mu_0^{-1}(t) \rightarrow U_{t_0}^* \times \mu_0^{-1}(t) \) is an isomorphism over \( \mu_0^{-1}(t) \). Hence \( \Phi'': Y_{a,t} \times (\mu_0^{-1}(t))_{\text{red}} \rightarrow U_{t_0}^* \times (\mu_0^{-1}(t))_{\text{red}} \) is an isomorphism over \((\mu_0^{-1}(t))_{\text{red}}\). Therefore we infer that \( \Phi: Y_{a,t} \rightarrow U_{t_0}^* \times t \) is a biholomorphic map for each \( t \in V_{t_0} \). From this it follows that the map \( \Phi: \mathcal{E} \rightarrow U_{t_0}^* \times V_{t_0} \) is a continuous bijective map. Furthermore, since \( \Phi \) is a simple analytic map, especially a proper map, it gives a homeomorphism between \( \mathcal{E} \) and \( U_{t_0}^* \times V_{t_0} \). So it suffices to show that a natural injective homomorphism \( \mathcal{E}_{t_0} \cap \mathcal{E}_{t_0} - \Phi_{*} \mathcal{E}_{t} \) of sheaves is surjective in order to prove that \( \Phi: \mathcal{E} \rightarrow U_{t_0}^* \times V_{t_0} \) is a biholomorphic map. We denote by \( \mathcal{M} \) the cokernel of the homomorphism \( \mathcal{E}_{t_0} \cap \mathcal{E}_{t_0} - \Phi_{*} \mathcal{E}_{t} \rightarrow \Phi_{*} \mathcal{E}_{t} \), i.e.,

\[ 0 \rightarrow \mathcal{E}_{t_0} \cap \mathcal{E}_{t_0} - \Phi_{*} \mathcal{E}_{t} \rightarrow \mathcal{E} \rightarrow \mathcal{M} \rightarrow 0 \] (exact).

We restrict this exact sequence of sheaves on the fiber \( U_{t_0}^* \times t \) for each \( t \in V_{t_0} \). Then we have
0 \rightarrow \mathcal{O}|_{\mathcal{U}_0} \rightarrow \Phi|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \rightarrow \mathcal{O}|_{\mathcal{U}_0} \rightarrow 0 \quad \text{(exact)}.

Well, since \( \Phi : \mathcal{O}_a \rightarrow \mathcal{U}_0 \times \mathcal{V}_0 \) is a finite map, \( \Phi|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \) is canonically isomorphic to \( (\Phi|_{\mathcal{U}_0})|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \). By this isomorphism we identify the sheaf \( \Phi|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \) with \( (\Phi|_{\mathcal{U}_0})|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \). Then the homomorphism \( \mathcal{O}|_{\mathcal{U}_0} \rightarrow \Phi|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \) in the above exact sequence is nothing but the one \( \mathcal{O}|_{\mathcal{U}_0} \rightarrow (\Phi|_{\mathcal{U}_0})|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \), which is an isomorphism, since \( \Phi : \mathcal{U}_0 \rightarrow \mathcal{U}_0 \times \mathcal{V}_0 \) is a biholomorphic map as seen above. Therefore we conclude that \( \mathcal{O}|_{\mathcal{U}_0} \rightarrow \mathcal{V}_0 \) for any \( t \in \mathcal{V}_0 \); hence by Nakayama's Lemma, we have \( \mathcal{O} = 0 \). This means the homomorphism \( \mathcal{O}|_{\mathcal{U}_0} \rightarrow \Phi|_{\mathcal{U}_0} \mathcal{O}|_{\mathcal{U}_0} \) is an isomorphism. Consequently \( \mathcal{O}_a \rightarrow \mathcal{U}_0 \times \mathcal{V}_0 \) is a biholomorphic map, so an isomorphism over \( \mathcal{V}_0 \) as desired. This completes the proof of the claim. Hence the sheaf \( \tilde{\mathcal{O}}(m) \) is coherent by the reason explained before.

**Step II:** We set \( \mathcal{X} = \text{Specan} \tilde{\mathcal{O}}(m) \) (analytic spectrum). In the subsequence we shall construct Specan \( \tilde{\mathcal{O}}(m) \) explicitly, and show that it has the properties (a) and (b) in the theorem. We take an open covering \( \{ \mathcal{U}_a \}_{a \in A} \) of \( \mathcal{X} \) which has the following properties:

(i) For each \( \alpha \in A \) there exist \( h_1^\alpha, \ldots, h_m^\alpha \in \mathcal{O}(\mathcal{U}_a) \tilde{\mathcal{O}}(m) \) such that \( (h_1^\alpha, \ldots, h_m^\alpha) \) generates \( \tilde{\mathcal{O}}(m) \) as \( \mathcal{O}(\mathcal{U}_a) \)-module at every point \( p \) in \( \mathcal{U}_a \) (this is possible, for the sheaf \( \tilde{\mathcal{O}}(m) \) is coherent);

(ii) For each \( \alpha \in A \) there is an isomorphism \( \phi_\mathcal{U}_a : \mathcal{U}_a \rightarrow U_\mathcal{V} \times \mathcal{V}_a \) where \( U_\mathcal{V} = \mathcal{U}_a \cap \mathcal{V}(\mathcal{U}_a) \), and \( \mathcal{V}_a \) and \( \mathcal{V}_\mathcal{V} = \omega(\mathcal{U}_a) \), such that the following diagram:

\[ \begin{array}{ccc}
\mathcal{U}_a & \rightarrow & U_\mathcal{V} \times \mathcal{V}_a \\
\downarrow v_\mathcal{V} & & \downarrow \pi_\mathcal{V} \\
\mathcal{V}_\mathcal{V} & \rightarrow & \mathcal{V}_a
\end{array} \]

is commutative (this is possible, for the family \( \omega : \mathcal{X} \rightarrow M \) is locally trivial by assumption).

We define a map \( h_\mathcal{V} : \mathcal{U}_a \rightarrow \mathcal{U}_a \times \mathcal{C}_m \) by

\[ h_\mathcal{V}(p) = (p, h_1^\alpha(p), \ldots, h_m^\alpha(p)) \]

for \( p \in \mathcal{U}_a \), where \( \mathcal{U}_a := \mathcal{U}_a \cap \mathcal{S}(\omega) \). We denote by \( \mathcal{U}_a^* \) the closure of the image \( h_\mathcal{V}(\mathcal{U}_a) \) in \( \mathcal{U}_a \times \mathcal{C}_m \) and by \( v_\mathcal{V} \) the restriction of the projection: \( \mathcal{U}_a \times \mathcal{C}_m \rightarrow \mathcal{U}_a \rightarrow \mathcal{U}_a^* \). Then \( \mathcal{U}_a^* \) is an analytic subvariety of \( \mathcal{U}_a \times \mathcal{C}_m \) and \( v_\mathcal{V} : \mathcal{U}_a^* \rightarrow \mathcal{U}_a^* \) is a finite map by the same reason as in Step I.

Note that the analytic restriction \( \tilde{\mathcal{O}}(m)|_{\mathcal{U}_a^*} \) of the sheaf \( \tilde{\mathcal{O}}(m) \) to \( \mathcal{U}_a^* \) is isomorphic to the sheaf \( \tilde{\mathcal{O}}_{\mathcal{U}_a^*} \) of germs of weakly holomorphic functions on \( \mathcal{U}_a^* \) for any \( t \in \mathcal{V}_a \), and that \( h_1^\alpha|_{\mathcal{U}_a^*}, \ldots, h_m^\alpha|_{\mathcal{U}_a^*} \) generate a stalk \( \tilde{\mathcal{O}}_{\mathcal{U}_a^*} \) at any point \( p \in \mathcal{U}_a^* \).

By this notice and recalling the way to construct a normal model of an analytic space ([26]), we observe that:

\[ v_\mathcal{V} : \mathcal{U}_a^* \rightarrow U_\mathcal{V}, \quad t \in \mathcal{V}_a = \pi_\mathcal{V}(\mathcal{U}_a) \]

is the normalization of \( \mathcal{U}_a^* \) for any point \( \mathcal{U}_a^* = v_\mathcal{V}(\mathcal{U}_a) \).

Furthermore, by the same arguments used in Step I to show the existence of the isomorphism \( \Phi : \mathcal{O}_a \rightarrow \mathcal{U}_0 \times \mathcal{V}_0 \) over \( \mathcal{W}_a = U_\mathcal{V} \times \mathcal{V}_a \), we can show that there exists an
isomorphism $\psi_\alpha: \mathcal{U}_\alpha^* \to U_\alpha^* \times V_\alpha$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{U}_\alpha^* & \xrightarrow{\psi_\alpha} & U_\alpha^* \times V_\alpha \\
\downarrow v_\alpha & & \downarrow v_1 \times id_v_\alpha \\
\mathcal{U}_\alpha & \xrightarrow{\phi_\alpha} & U_\alpha \times V_\alpha
\end{array}
$$

is commutative, where $U_\alpha^* := v_\alpha^{-1}(U_\alpha)$. In view of the fact that each $h_\beta^f$ can be written as

$$h_\beta^f = \sum_{j=1}^{m_\beta} a_{ij}^f \cdot h_\beta^f$$

on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ for $1 \leq i \leq m_\alpha$, where $a_{ij}^f$ are holomorphic functions there, we can define a map $\tilde{\phi}_{\alpha\beta}: (\mathcal{U}_\beta \cap \mathcal{U}_\alpha) \times C^{m_\beta} \to (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times C^{m_\alpha}$ by

$$\tilde{\phi}_{\alpha\beta}(p, z^\beta) = (p, \sum_{j=1}^{m_\beta} a_{ij}^f(p) z_j^\beta, \ldots, \sum_{j=1}^{m_\beta} a_{m_\beta j}^f(p) z_j^\beta)$$

for $(p, z^\beta) \in (\mathcal{U}_\beta \cap \mathcal{U}_\alpha) \times C^{m_\beta}$. It is clear that $\tilde{\phi}_{\alpha\beta}(\mathcal{U}_\beta^*) \subset \mathcal{U}_\alpha^*$, for $\alpha, \beta \in A$, where $\mathcal{U}_\alpha^* := v_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. We denote by $\phi_{\alpha\beta}: \mathcal{U}_\beta^* \to \mathcal{U}_\alpha^*$ the restriction of $\tilde{\phi}_{\alpha\beta}$ to $\mathcal{U}_\alpha^*$. Since $h_\beta^f \cdot h_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, one has $\phi_{\beta\alpha} \circ \phi_{\alpha\beta} = \text{identity on } v_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$; hence $\phi_{\alpha\beta} \circ \phi_{\beta\alpha} = \text{identity on } \mathcal{U}_\alpha^*$ for $v_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is an open dense subset of $\mathcal{U}_\alpha^*$ for every $\alpha, \beta \in A$. From this it follows that each $\phi_{\alpha\beta}$ is a biholomorphic map.

For any triple $\alpha, \beta, \gamma \in A$, we have

(a) $\phi_{\beta\alpha}(\mathcal{U}_\beta^* \cap \mathcal{U}_\alpha^*) \subset \mathcal{U}_\alpha^* \cap \mathcal{U}_\gamma^*$,

(b) $\phi_{\alpha\gamma}(\mathcal{U}_\alpha^* \cap \mathcal{U}_\gamma^*) = (\phi_{\beta\alpha}(\mathcal{U}_\alpha^* \cap \mathcal{U}_\beta^*)) \circ (\phi_{\beta\gamma}(\mathcal{U}_\beta^* \cap \mathcal{U}_\gamma^*))$.

The condition (a) is obviously satisfied, and the condition (b) follows from the facts that $\phi_{\alpha\beta} \circ v_\beta^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) = h_\beta^f h_\alpha^{-1}$ and $v_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is an open dense subset of $\mathcal{U}_\beta^*$ for every $\alpha, \beta \in A$. Therefore we can glue up the family $\{\mathcal{U}_\alpha^*\}_{\alpha \in A}$ by the family $\{\phi_{\alpha\beta}\}_{\alpha, \beta \in A}$ of biholomorphic maps, and obtain an analytic variety $\mathcal{X}$, which is nothing but Specan $\hat{\mathcal{O}}_{\mathcal{X}}(\mathfrak{m})$. Furthermore the following diagram

$$
\begin{array}{ccc}
\mathcal{U}_\alpha^* & \xrightarrow{\phi_{\alpha\beta}} & \mathcal{U}_\beta^* \\
\downarrow v_\alpha & & \downarrow v_{\beta|\gamma} \\
\mathcal{U}_\alpha \cap \mathcal{U}_\beta & \xrightarrow{\mathfrak{m}^{-1}} & \mathcal{U}_\beta \cap \mathcal{U}_\gamma
\end{array}
$$

is commutative. This follows from the facts that $v_\alpha = h_\alpha^{-1}$ on $v_\alpha^{-1}(\mathcal{U}_\alpha)$, $\phi_{\alpha\beta}(v_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)) = h_\beta^f h_\alpha^{-1}$, and $v_\beta^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ is an open dense subset of $\mathcal{U}_\beta^*$ for $\alpha, \beta \in A$. So the family $\{v_\alpha\}_{\alpha \in A}$ defines a holomorphic map from $\mathcal{X}$ onto $\mathcal{X}$, which we denote by $v$. We set $\mathfrak{m} := \mathfrak{m} \circ v: \mathcal{X} \to M$.

Then by (10.1), $v_t: X_t \to Z_t$ is the normalization of $Z_t$ for any $t \in M$ ($X_t := \pi_t^{-1}(t)$, $Z_t := \mathfrak{m}^{-1}(t)$, $v_t := v_{1|X_t}: X_t \to Z_t$), and by (10.2), the family $\pi: \mathcal{X} \to M$ is locally trivial. Thus the existence of a family $\mathcal{X} \to M$ which has the properties $(\alpha), (\beta)$ in the theorem has been proved.

**Step III:** Finally, we shall prove the uniqueness of the family $\pi: \mathcal{X} \to M$ and the surjective holomorphic map $v: \mathcal{X} \to \mathcal{X}$ constructed in Step II up to isomorphisms over $M$. Let $\pi_1: \mathcal{X}_1 \to M$ be another family of analytic varieties parametrized by the analytic variety $M$, and $v_1: \mathcal{X}_1 \to \mathcal{X}$ another surjective holomorphic map over $M$, which have the properties $(\alpha), (\beta)$ in the theorem. For $t \in M$, $X_t := \pi_t^{-1}(t)$ and $X_1(t) := \pi_1^{-1}(t)$ are the normalization of $Z_t := \mathfrak{m}^{-1}(t)$. Hence there exists a unique biholomorphic map $\Phi_1$:
$X_t \rightarrow X_t^{(1)}$. The family $\Phi := \{\Phi_t\}_{t \in M}$ of maps define a bijective map from $\mathcal{Z}$ onto $\mathcal{Z}_1$ over $\mathcal{Z}$, i.e., the following diagram

is commutative.

We will show that the map $\Phi$ is in fact a biholomorphic map. After taking refinement we may assume that the open covering $\{\mathcal{U}_a\}_{a \in A}$ of $\mathcal{Z}$, choosen in the proof of Step II, has the following property besides the properties (i), (ii):

(iii) $v_1$ is trivial over each $\mathcal{U}_a$, i.e., there exists an isomorphism $\psi_1^{(1)} : \mathcal{U}_a^{(1)} := v_1^{-1}(\mathcal{U}_a) \rightarrow U_a^{(1)} \times V_a$, where $U_a^{(1)} = v_1^{-1}(\mathcal{U}_a) \cap X_a^{(1)}$ for $p_a \in \mathcal{U}_a$ and $V_a = w(\mathcal{U}_a)$, such that the following diagram

is commutative.

Note that it follows that $v_1, \pi(t) : U_a^{(1)} \rightarrow U_a$ is the normalization of $U_a$ from the assumption that $v_1, \pi(t) : X_t^{(1)} \rightarrow Z_t$ is the one of $Z_t$ for any $t \in M$. Comparing (10.2) with (10.3), we observe that, by the uniqueness of a normalization up to isomorphisms, there exists an isomorphism $\mu_a : U_a^{(1)} \rightarrow U_a^{(1)}$ over $U_a$, so an isomorphism $\mu_a \times id_{V_a} : U_a^{(1)} \times V_a \rightarrow U_a^{(1)} \times V_a$ over $U_a \times V_a$. Then $(\psi_a^{(1)})^{-1} \circ (\mu_a \times id_{V_a}) \circ \psi_a$ give a biholomorphic map from $\mathcal{U}_a^{*}$ onto $\mathcal{U}_a^{*}$ over $\mathcal{U}_a$. Since $U_a^{(1)} := \pi^{-1}(t) \cap \mathcal{U}_a^{*}$ and $U_a^{(1)} := \pi^{-1}(t) \cap \mathcal{U}_a^{*}$ are the normalizations of $U_{a,t} := w^{-1}(t) \cap \mathcal{U}_a$ for any $t \in V_a$, a biholomorphic map from $U_a^{(1)}$ onto $U_a^{(1)}$ over $U_{a,t}$ is uniquely determined. Therefore we have

$$\Phi_{t | U_a^{*}} = (\psi_a^{(1)})^{-1} (\mu_a \times id_{V_a}) \circ \psi_a | U_a^{*}$$

for any $t \in V$ and $a \in A$; hence $\Phi_{t | U_a^{*}} = (\psi_a^{(1)})^{-1} (\mu_a \times id_{V_a}) \circ \psi_a$ for any $a \in A$. Consequently, $\Phi$ is surely a biholomorphic map. This shows that a family $\pi : \mathcal{Z} \rightarrow M$ and a holomorphic map $v : \mathcal{Z} \rightarrow \mathcal{Z}$ over $M$ which have the properties $(\alpha), (\beta)$ in the theorem are uniquely determined up to isomorphisms over $M$.

Q.E.D.

**Corollary (10.1):** Let $(Y \times M, \mathcal{Z}, w, M, 0)$ be a family of locally trivial displacements of an irreducible analytic subvariety $Z$ with ordinary singularities of a complex manifold $Y$ parametrized by an analytic variety $M$. Let $v : X \rightarrow Z$ be the normalization of $Z$, and let $f : X \rightarrow Y$ be the composite map of the normalization $v : X \rightarrow Z$ of $Z$ and the inclusion map $i : Z \subseteq Y$. (Note that $f : X \rightarrow Y$ is a locally stable
holomorphic map by the definition of an analytic subvariety with ordinary singularities (cf. Definition (6.1)). Then there exists a family \((\mathcal{X}, F, \pi, M, 0)\) of deformations of \(f: X \to Y\) such that \((Y \times M, F(\mathcal{X}), \pi', M, 0) = (Y \times M, \mathcal{Z}', \pi, M, 0)\), where \(\pi'\) denotes the restriction of the canonical projection \(\text{Pr}_M: Y \times M \to M\) to \(F(\mathcal{X})\). Furthermore, such a family \((\mathcal{X}, F, \pi, M, 0)\) is uniquely determined up to isomorphisms of families of deformations of a holomorphic map (for definition see [8]).

**Proof**: This follows immediately from Theorem (10.1). Q. E. D.

§11. Main theorem and concluding results

1°) In this section we shall unify the results obtained till now and derive some concluding results. To state our main theorem we introduce some notations and terminology. For families of deformations of holomorphic maps the concepts of “morphism”, “equivalence”, “completeness”, “universality”, e.t.c. are defined ([8], [23], [24]). We shall give the definitions of corresponding ones for families of locally trivial displacements of irreducible analytic subvarieties with ordinary singularities in a compact complex manifold.

Let \(f: X \to Y\) be a locally stable holomorphic map between complex manifolds with \(X, Y\) compact and \(\dim X < \dim Y\), and let \(Z := f(X)\) be an irreducible analytic subvariety with ordinary singularities in \(Y\) which is the image of \(X\) by \(f\).

**Definition (11.1)**: Let \(E = (Y \times M, \mathcal{Z}, \pi, M, 0)\) be a family of locally trivial displacements of \(Z\) in \(Y\), let \(M'\) be an analytic variety, and let \(h: M' \to M\) be a holomorphic map with \(h(0') = 0\), where \(0'\) is an assigned point in \(M'\). We define the family \(h^*E = (Y \times M', \mathcal{Z}', \pi', M', 0')\) induced by \(h\) as follows:

(i) \(\mathcal{Z}' = \mathcal{Z} \times_M M'\),

(ii) \(\pi' = \text{Pr}_M: \mathcal{Z}' \to M'\).

In particular, if \(M'\) is a subvariety of \(M\) and if \(h: M' \to M\) is the natural injection, then we call the family \(h^*E\) the restriction to \(M'\) and denote it by \(E|_{M'}\).

**Definition (11.2)**: Let \(E = (Y \times M, \mathcal{Z}, \pi, M, 0)\) and \(E' = (Y \times M', \mathcal{Z}', \pi', M', 0')\) be two families of locally trivial displacements of \(Z\) in \(Y\). By a morphism (resp. isomorphism) from \(E\) into \(E'\) we mean a holomorphic map (resp. biholomorphic map) \(h: M \to M'\) with \(h(0) = 0'\), such that \(h^*E' = E\).

**Definition (11.3)**: If there exists an isomorphism from \(E\) onto \(E'\), we say that \(E\) and \(E'\) are equivalent.

**Definition (11.4)**: We say a family \(E = (Y \times M, \mathcal{Z}, \pi, M, 0)\) of locally trivial displacements of \(Z\) in \(Y\) is maximal at \(0\) if for any family \(E' = (Y \times M', \mathcal{Z}', \pi', M', 0')\) of locally trivial displacements of \(Z\) in \(Y\), there exists an open neighborhood \(N'\) of \(0'\) in \(M'\) and a morphism from the restriction \(E'|_{N'}\) of \(E'\) over \(N'\) into \(E\).

**Definition (11.5)**: We say a family \(E = (Y \times M, \mathcal{Z}, \pi, M, 0)\) of locally trivial displacements of \(Z\) in \(Y\) is universal at \(0\) if it satisfies the following:

(i) \(E\) is maximal at \(0\);

(ii) for any family \(E' = (Y \times M', \mathcal{Z}', \pi', M', 0')\) of locally trivial displacements of \(Z\) in \(Y\), let \(h: N' \to M\) be a holomorphic map from an open neighborhood \(N'\) of \(0'\) into \(M\)
which give rise to a morphism from $E'_{|N'}$ into $E$. Then the germ of such a holomorphic map $h$ at $0'$ is uniquely determined.

We are really interested only in \textit{germs of families}. So a family $E = (Y \times M, \mathcal{Z}, \pi, M, 0)$ is identified with its restriction to any open neighborhood of $0$ in $M$, and we consider that two families $E = (Y \times M, \mathcal{Z}, \pi, M, 0)$ and $E' = (Y \times M', \mathcal{Z}', \pi', M', 0')$ are isomorphic if their restrictions to some open neighborhoods of $0$ and $0'$, respectively, are isomorphic. Note that the concepts of \textit{maximal at $0$}, or \textit{universal at $0$} defined above have senses when we pass over to \textit{germs of families}.

It is clear that the set of germs of families of deformations of the holomorphic map $f : X \rightarrow Y$ (resp. the set of germs of families of locally trivial displacements of $Z$ in $Y$) forms a category, which we denote by $\mathcal{D}(f)$ (resp. $\mathcal{L}(Z)$). With these notations and terminology we have the following:

\textbf{Theorem (11.1):} There exists canonically a functor $\mathcal{F} : \mathcal{D}(f) \rightarrow \mathcal{L}(Z)$ such that:

(i) $\mathcal{F}$ is surjective;

(ii) let $D, D'$ be any objects of $\mathcal{D}(f)$. Then $D$ is equivalent to $D'$ if, and only if, $\mathcal{F}(D)$ is equivalent to $\mathcal{F}(D')$;

(iii) $D \in \text{Ob}(\mathcal{D}(f))$ (the objects of the category $\mathcal{D}(f)$) is complete if, and only if, $\mathcal{F}(D)$ is maximal;

(iv) $D_0 \in \text{Ob}(\mathcal{D}(f))$ is universal if, and only if, $\mathcal{F}(D_0)$ is universal.

\textit{Proof:} We define a functor $\mathcal{F}$ by $\mathcal{F}(D) = (Y \times M, F(\mathcal{Z}), \varpi, M, 0)$ ($\varpi = \pi_{M, F(\mathcal{Z})}$; $F(\mathcal{Z}) \rightarrow M$) for $D = (\mathcal{Z}, F, \pi, M, 0) \in \text{Ob}(\mathcal{D}(f))$. $\mathcal{F}(D)$ is surely an object of the category $\mathcal{L}(Z)$ by Proposition (9.2). The surjectivity of the functor $\mathcal{F}$ follows from Corollary (10.1).

For any $D_1, D_2 \in \text{Ob}(\mathcal{D}(f))$ it follows from definition that if $D_1$ is equivalent to $D_2$, then $\mathcal{F}(D_1)$ is equivalent to $\mathcal{F}(D_2)$. Conversely, if $L = (Y \times M, \mathcal{Z}, \varpi, M, 0)$, $L' = (Y \times M', \mathcal{Z}', \varpi', M', 0') \in \text{Ob}(\mathcal{L}(Z))$ are equivalent, then there exists (a germ of) a biholomorphic map $h : M' \rightarrow M$ with $h(0) = 0'$, such that $h^*L' = L$. Let $D = (\mathcal{Z}, F, \pi, M, 0)$, $D' = (\mathcal{Z}', F', \pi', M', 0')$ be objects of $\mathcal{D}(f)$ such that $\mathcal{F}(D) = L$ and $\mathcal{F}(D') = L'$, respectively. We denote by $h^*D' = (\mathcal{Z}''$, $F''$, $\pi''$, $M$, 0) the family induced by $h : M \rightarrow M'$ (i.e. $\mathcal{Z}'' : = \mathcal{Z} \times _M M$, $F'' : = F' \times id_M$; $\mathcal{Z}' \times _MM \rightarrow (Y \times M') \times _M M = Y \times M$, $\pi'' : = Pr_M$; $\mathcal{Z}'' \times _MM \rightarrow M$). Then, since $h^*L' = L, h^*D'$ and $D$ give rise to the commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\pi} & \mathcal{Z}' \\
\downarrow v & & \downarrow w' \\
M & & M \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{Z} & \xrightarrow{\pi} & \mathcal{Z}' \\
\downarrow v & & \downarrow w \\
M & & M \\
\end{array}
\]

which have the properties $(\alpha)$ and $(\beta)$ in Theorem (10.1). By just this theorem such commutative diagrams are uniquely determined up to isomorphisms over $M$ for a family $w : \mathcal{Z} \rightarrow M$. Therefore we infer that $h^*D'$ and $D$ are equivalent. On the other hand, it is obvious that $h^*D'$ and $D'$ are equivalent. Consequently, $D$ and $D'$ are equivalent.

The assertions (iii) and (iv) follows immediately from Theorem (10.1) and definition.

Q. E. D.
Proposition (11.1): For an irreducible analytic subvariety $Z$ with ordinary singularities of a compact complex manifold $Y$, there exists a family $(Y \times M, \mathcal{Z}, \sigma, M, 0)$ of locally trivial displacements of $Z$ in $Y$ such that:

(i) the characteristic map $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ (Definition (8.3)) is injective;
(ii) the family is maximal at any point $t \in M$;
(iii) the family is universal at 0.

Furthermore, if $H^1(Z, \mathcal{N}_{Z/Y})=0$, then $M$ is non-singular and $\sigma_0: T_0(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ is bijective.

Proof: Let $f: X \to Y$ be a locally stable holomorphic map such that $f(X)=Z$. Note that by Theorem (11.1), there is an isomorphism between the category of germs of families of deformations of the holomorphic map $f: X \to Y$ and the one of germs of locally trivial displacements of $Z$ in $Y$. Furthermore, note that their characteristic maps are related as stated in Proposition (9.3). Then, restating K. Miyajima’s existence theorem of Kuranishi family for deformations of holomorphic maps ([23]) in terms of families of locally trivial displacements of $Z$ in $Y$, we obtain the above proposition except the assertion (iii). Since $f$ is locally stable, it is a finite map by Corollary (4.1); especially a non-degenerate map. Therefore the assertion (iii) follows from Theorem 2 in [24].

Q. E. D.

Proposition (11.2): Let $Z$ be an irreducible hypersurface with ordinary singularities in a compact complex manifold $Y$. Then there exists a family $(Y \times M, \mathcal{Z}, \sigma, M, 0)$ of locally trivial displacements of $Z$ in $Y$ such that:

(i) the characteristic map $\sigma_t: T_t(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ is injective for any point $t \in M$;
(ii) the family is universal at any point $t \in M$.

Furthermore, if $H^1(Z, \mathcal{N}_{Z/Y})=0$, then $M$ is non-singular and $\sigma_t: T_t(M) \to H^0(Z, \mathcal{N}_{Z/Y})$ is bijective for $t \in M$ sufficiently close to 0.

Proof: Let $(Y \times M', \mathcal{Z}, \sigma, M', 0)$ be the family of locally trivial displacements of $Z$ in $Y$ which has the properties (i), (ii), (iii) in Proposition (11.1). We will show that this family has in fact the above properties (i), (ii) if we shrink $M'$ sufficiently small around 0.

First we show that if $Z$ is a hypersurface, the sheaf $\mathcal{N}_{Z/Y}$ is canonically isomorphic to a subsheaf of the invertible sheaf $\mathcal{O}_Z([Z]_{Z})$, where $[Z]_Z$ denotes the restriction to $Z$ of the line bundle $[Z]$ determined by the divisor $Z$, and $\mathcal{O}_Z([Z]_{Z})$ denotes the sheaf of germs of local cross-sections of $[Z]_{Z}$. We define a homomorphism $d: \Theta_Y \to \mathcal{O}_Z([Z]_{Z})$ of sheaves as follows: Let $q$ be a point in $Y$, $\theta_q$ an element of the stalk $\Theta_{Y,q}$ and $\theta$ a local cross-section of $\Theta_Y$ in an open neighborhood $U \subset Y$ of $q$, which represents $\theta_q$ at $q$. Let $h=0$ be a defining equation of $Z$ in $U$. Then we define a stalk map $d_q: \Theta_{Y,q} \to \mathcal{O}_Z([Z]_{Z})$ by $\theta_q \mapsto (\theta h_q)|_{\Theta_Y}$, where $\partial h$ denotes the differentiation of $h$ by $\theta$. It is obvious this stalk map defines a homomorphism $d: \Theta_Y \to \mathcal{O}_Z([Z]_{Z})$ of sheaves. The kernel of the homomorphism $d$ is $\Theta_Y(log Z)$ by the definition of this sheaf. Therefore, the image of $d$ is isomorphic to the sheaf $\mathcal{N}_{Z/Y}=\Theta_Y/\Theta_Y(log Z)$.

Next we clarify how the characteristic map $\sigma_t: T_t(M') \to H^0(Z, \mathcal{N}_{Z/Y})$ is represented under the identification $\mathcal{N}_{Z/Y}=Im d$. Let $(\mathcal{U}_i)_{i=1}^n$ be the open covering of $Y \times M'$ that has been taken to define the characteristic map $\sigma$ in §8. In the following
we use the same notations as there. Let \( h_i(y_t) = 0 \) be a defining equation of \( Z \) in \( U_t = \mathfrak{A}_t \cap (Y \times 0) \). We set \( h_{i,t'}(y_t) := h_i(Y(t, t')) \) and \( \theta_{i,t'} = \sum_{a=1}^d \frac{\partial Y_a}{\partial t_{i,t'}} \left( -\frac{\partial}{\partial y^a_t} \right) \) for \( t' \in M' \) sufficiently close to 0. Then \( h_{i,t'}(y_t) = 0 \) is a defining equation of \( Z_{t'} \) in \( \mathfrak{A}_t \cap (Y \times t') \).

Hence for any \( \left( \frac{\partial}{\partial t} \right)_{t=t'} \in T_{t'}(M') \), we have

\[
\sigma_t \left( \left( \frac{\partial}{\partial t} \right)_{t=t'} \right) = \{ \theta_{i,t'} h_{i,t'}(y_t) \}_{i,t'}
\]

\[
= \left\{ \frac{\partial}{\partial t'} h(Y(t, t')) \right\}_{t=t'} \}
\]

By using this description of the characteristic map \( \sigma_t \), we can show that it is injective for any point \( t \) in a sufficiently small open neighborhood \( M \subset M' \) of 0 as follows:

Suppose to the contrary, there exists a sequence \( \{ t_n \} \) of points in \( M' \) converging to 0 such that for each \( n \) there exists an element \( v_n = \sum_{a=1}^d v_n^a \left( \frac{\partial}{\partial t_{i=n}} \right) \in T_{t_n}(M') \) with \( v_n \not= 0 \) and \( \sigma_{t_n}(v_n) = 0 \). (Here we identify \( T_{t_n}(M') \) with the subspace \( \{ v \in T_{t_n}(C') \mid v(\mathcal{J}(M'), t_n) = 0 \} \) of \( T_{t_n}(C') \), where we consider \( M' \) is an analytic subvariety of an open subset of \( C' \), and \( \mathcal{J}(M', t_n) \) denotes the stalk at \( t_n \) of the ideal sheaf of \( M' \) in \( \mathcal{O}_{C'} \).) Note that the fact \( \sigma_{t_n}(v_n) = 0 \) is equivalent to

\[
v_n \cdot h_i(Y(t, t'))_{t=t_n} = \sum_{a=1}^d v_n^a \left( \frac{\partial Y_a}{\partial t_{i=t_n}} \right) \cdot \frac{\partial h_i}{\partial y^a_t}(Y(t, t')) \equiv 0
\]

on \( Z_t \cap \mathfrak{A}_t \), i.e., the locus \( h_{i,t_n} = 0 \), for every \( i \in I \). We may assume that \( \| v_n \| := \sqrt{\sum_{a=1}^d (v_n^a)^2} = 1 \), so the sequence \( \{ v_n \} \) of tangent vectors converges to a non-zero vector \( v_0 \) of \( T_0(M) \) when \( n \) tends to \( \infty \). Then we have \( v_0 \cdot h_i(Y(t, t'))_{t=0} \equiv 0 \) on \( Z \cap \mathfrak{A}_t \) for every \( i \in I \). This means \( \sigma_0(v_0) = 0 \), which is a contradiction to the fact that \( \sigma_0 \) is injective.

By the facts that the characteristic map \( \sigma_t \) is injective at any point \( t \) in a sufficiently small open neighborhood \( M \subset M' \) of 0, and that the family is maximal at any point \( t \) in \( M' \), we infer that it is universal at any point \( t \in M \) by Theorem 2 in [24].

Finally, we show that the latter half part of the proposition. If \( H^1(Z, \mathcal{N}_{Z/Y}) = 0 \), then \( M \) is non-singular by Proposition (11.1). To show that the characteristic map \( \sigma_t : T_t(M) \rightarrow H^0(Z_t, \mathcal{N}_{Z/t}) \) is bijective, we introduce a sheaf \( \mathcal{N}_{Z/Y} := \Theta_{Y \times M} / \Theta_{Y \times M}(\log \mathcal{Z}) \). Since the family \( \mathcal{Z} \rightarrow M \) is locally trivial, we may consider that \( \mathcal{Z} = Z \times M \subset Y \times M \) and \( \mathcal{N}_{Z/Y} = \pi^*_Z \mathcal{N}_{Z/t} \) locally, so we infer that \( \mathcal{N}_{Z/Y} |_{Z_t} = \mathcal{N}_{Z/t} \) for any point \( t \in M \), and that the sheaf \( \mathcal{N}_{Z/Y} \) is flat over \( M \). Then by the upper semicontinuity of cohomology ([2], Theorem (4.12)), we have \( \dim H^0(Z_t, \mathcal{N}_{Z/t}) \leq \dim H^0(Z, \mathcal{N}_{Z/t}) \) for \( t \in M \) sufficiently close to 0. On the other hand, the characteristic map \( \sigma_t : T_t(M) \rightarrow H^0(Z_t, \mathcal{N}_{Z/t}) \) is bijective at \( t = 0 \) by Proposition (11.1), and injective at any point \( t \in M \) as showed above; hence \( \sigma_t \) is in fact bijective at \( t \in M \) sufficiently close to 0.

Q. E. D.

Remark (11.1): K. Kodaira proved in [10] that if \( Z \) is a "semi-regular" surface with ordinary singularities in a threefold \( Y \), then there exists a family \( (Y \times M, \mathcal{Z}, \omega, M, 0) \) of locally trivial displacements of \( Z \) in \( Y \) with \( M \) non-singular, which has the following properties:

-
(i) the characteristic map $\sigma_t: T_0(M)\to H^0(Z, \mathcal{N}_{Z,iY})$ is bijective for any point $t \in M$;
(ii) the family is maximal at 0.

If $Y$ is a threefold subject to the conditions $H^2(Y, \sigma_Y)=0$; for example, if $Y$ is birationally equivalent to $\mathbb{P}^1(C)$, the semi-regularity condition on $Z$ is equivalent to $H^1(Z, \mathcal{N}_{Z,iY})=0$. Hence Kodaira's existence theorem follows from ours. Generally there is a gap between Kodaira's theorem and ours for surfaces.

**Proposition (11.3):** Let $(Y \times M, \mathcal{Z}, w, M, 0)$ be a family of locally trivial displacements of an irreducible analytic subvariety $Z$ with ordinary singularities of a compact complex manifold $Y$ with $M$ non-singular.

Suppose the characteristic map $\sigma_0: T_0(M)\to H^0(Z, \mathcal{N}_{Z,iY})$ is surjective, then the family is maximal at 0.

**Proof**: This proposition is obtained by restating the following E. Horikawa's theorem (Theorem (2.1) in [8]) in terms of families of locally trivial displacements of $Z$ in $Y$: if the characteristic map $\tau_0: T_0(M)\to H^0(X, \mathcal{F}_{X,iY})$ is surjective for a family of deformations of a non-degenerate holomorphic map $f: X\to Y$, then it is complete at 0.

Q. E. D.

**Remark (11.2):** K. Kodaira proved in [10] that if $Z$ is a "semi-regular" surface, then the above holds.

**Appendix:** Stability of a simultaneously infinitesimally stable multi-germ of a holomorphic map

0°) In this appendix we shall prove the stability of a simultaneously infinitesimally stable multi-germ of a holomorphic map, which is a complex analytic analogue of a well known fact in the theory of stable $C^\infty$ mappings.

The stability of a simultaneously infinitesimally stable multi-germ of a holomorphic map has two aspects. One is that any unfolding of a simultaneously infinitesimally stable multi-germ of a holomorphic map is trivial (Theorem (A)). Though another is slightly more complicated to explain, we venture to say that for any given simultaneously infinitesimally stable multi-germ $f: (X, S)\to (Y, q)$ of a holomorphic map, every holomorphic map $g$ defined in an open neighborhood of $S$ in $X$ "sufficiently close" to $f: (X, S)\to (Y, q)$ always determines the multi-germ $g: (X, S')\to (Y, q')$ that is isomorphic to $f: (X, S)\to (Y, q)$, where $S' \subset X$ are finite subsets close to $S$ (Theorem (B)). This fact was proved by A. I. Arnold in [1] by a different method from ours. These two stability of a simultaneously infinitesimally stable multi-germ of a holomorphic map play essential roles in the study of locally stable holomorphic maps as we have seen in this paper. We use formal power series expansion, and to prove the convergence of formal solutions we adopt the techniques of K. Kodaira and D. C. Spencer in [13].

Now we explain notations which will be frequently used in the subsequence. Let $\mathbb{C}^n$ and $\mathbb{C}^m$ be complex affine spaces of dimension $n$ and $m$, respectively. We denote by $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ linear coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. An open polydisc in $\mathbb{C}^n$ is a subset $\Delta(n; p; r) \subset \mathbb{C}^n$ of the form
\[ D(n; p; r) = \{ x \in \mathbb{C}^n \mid |x_j - x_j(p)| < r \text{ for } 1 \leq j \leq n \} \]

The point \( p \in \mathbb{C}^n \) is called the center of the polydisc, and \( r \in \mathbb{R} (r > 0) \) is called the radius. The closure of \( D(n; p; r) \) will be called the closed polydisc with center \( p \) and radius \( r \), and will be denoted by \( \overline{D}(n; p; r) \). If the center \( p = 0 \), then we write \( D(n; r) \) and \( \overline{D}(n; r) \) instead of \( D(n; 0; r) \) and \( \overline{D}(n; 0; r) \) for simplicity.

For any analytic coherent sheaf \( \mathscr{F} \) and any compact subset \( K \) of \( \mathbb{C}^n \), we denote by \( \mathscr{F}(K) \) the module of cross-sections of the sheaf \( \mathscr{F} \) defined on open neighborhoods of \( K \).

If a given analytic coherent sheaf \( \mathscr{F} \) is canonically isomorphic to a free sheaf \( \Theta_{\mathbb{C}^n} \), i.e., the direct sum of \( l \) copies of the sheaf \( \Theta_{\mathbb{C}^n} \), in an open neighborhood of \( K \), we identify \( \mathscr{F} \) with \( \Theta_{\mathbb{C}^n}^l \) via the canonical isomorphism in an open neighborhood of \( K \), and we define a norm \( \| s \|_K \) by

\[ \| s \|_K = \max \{ \sup_{1 \leq i \leq l} |s_i(x)| \mid x \in K \} \]

for \( s = (s_1, \ldots, s_l) \in \mathscr{F}(K) = \Theta_{\mathbb{C}^n}(K) \). Let other notations be as in Chapter I, §1.

1°) Lemma (A): Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) be an infinitesimally stable germ of a holomorphic map at \( 0 \in \mathbb{C}^n \). We suppose that \( tf(\Theta_{\mathbb{C}^n, 0}) = f^*\Theta_{\mathbb{C}^n, 0} \). Then for each \( i = 1, \ldots, n \), there exists a Weierstrass polynomial \( P_i(x_0, y_1, \ldots, y_m) \in \Theta_{\mathbb{C}^n, 0}[x_i] \) in \( x_i \) with coefficients in \( \Theta_{\mathbb{C}^n, 0} \) such that

\[ P_i(x_0, f(x)) \cdot x_0^{\lambda_0} x_1^{\lambda_1} \cdots x_i^{\lambda_i - 1} \cdots x_m^{\lambda_m} f^* \left( \frac{\partial}{\partial y_j} \right) \in tf(\Theta_{\mathbb{C}^n, 0}) \]

for \( \lambda_0 = 0, 0 \leq \lambda_1 \leq d_1 - 1, \ldots, 0 \leq \lambda_i - 1 \leq d_i - 1, 1 \leq j \leq m \), where we set \( x_0 = 1 \) to unify the definition, and \( d_1 \) denotes the degree of \( P_i(x_0, y_1, \ldots, y_m) \) with respect to \( x_0 \) for \( 1 \leq i' \leq i - 1 \).

Proof: Assume that the lemma is proved for all natural numbers \( i \), and we shall prove it for \( i \geq 1 \). By the assumption that \( f \) is infinitesimally stable at 0, for every \( x_0, x_1^{\lambda_0} x_1^{\lambda_1} \cdots x_i^{\lambda_i - 1} \cdots x_m^{\lambda_m} f^* \left( \frac{\partial}{\partial y_j} \right) \in f^* \Theta_{\mathbb{C}^n, 0} \), there exist \( \xi_{\lambda_0, \lambda_1, \ldots, \lambda_i, i} \in \Theta_{\mathbb{C}^n, 0} \) and \( \zeta_{\lambda_0, \lambda_1, \ldots, \lambda_i, i} \in \Theta_{\mathbb{C}^n, 0} \) such that

\[ x_0, x_1^{\lambda_0} x_1^{\lambda_1} \cdots x_i^{\lambda_i - 1} \cdots x_m^{\lambda_m} f^* \left( \frac{\partial}{\partial y_j} \right) = t f(\xi_{\lambda_0, \lambda_1, \ldots, \lambda_i, i}) + \omega f(\zeta_{\lambda_0, \lambda_1, \ldots, \lambda_i, i}). \]

Hence, representing \( \xi_{\lambda_0, \lambda_1, \ldots, \lambda_i, i} \) as

\[ \xi_{\lambda_0, \lambda_1, \ldots, \lambda_i, i}(y) = \sum_{k=1}^{m} a_{\lambda_0, \lambda_1, \ldots, \lambda_i, i, k}(y) \left( \frac{\partial}{\partial y_k} \right), \]

where \( a_{\lambda_0, \lambda_1, \ldots, \lambda_i, i, k}(y) \in \Theta_{\mathbb{C}^n, 0} \), one has

\[ x_0, x_1^{\lambda_0} x_1^{\lambda_1} \cdots x_i^{\lambda_i - 1} f^* \left( \frac{\partial}{\partial y_j} \right) = \sum_{k=1}^{m} a_{\lambda_0, \lambda_1, \ldots, \lambda_i, i, k}(y) \left( \frac{\partial}{\partial y_k} \right) \]

\[ = t f(\xi_{\lambda_0, \lambda_1, \ldots, \lambda_i, i}). \]

We rewrite this equality in the form...
\[
\sum_{\mu_0=0}^{d_0-1} \sum_{\mu_1=0}^{d_1-1} \cdots \sum_{\mu_{l-1}=0}^{d_{l-1}-1} \prod_{k=1}^{m} \{ \delta_{\lambda_0 \mu_0} \cdots \delta_{\lambda_1 \mu_1} \cdots \delta_{\lambda_{l-1} \mu_{l-1}} \cdot \delta_{j_k, \lambda_{l-1}} \cdot x_{i_k} - \delta_{\lambda_0 \mu_0} \cdots \delta_{\lambda_{l-1} \mu_{l-1}} \cdot \delta_{j_k, \lambda_{l-1}} \cdot f(x) \} \cdot x_{i_0}^x x_{i_1}^{x_1} \cdots x_{i_{l-1}}^{x_{l-1}} f^{*} \left( \frac{\partial}{\partial y_k} \right) \\
= tf \left( \xi_{\lambda_0 \lambda_1 \cdots \lambda_{l-1}, j_k} \right),
\]

where we set \( d_0 = 1 \) to unify the notations. We arrange all multi-indices \((\mu_0, \mu_1, \ldots, \mu_{l-1}, j_k)\) for \( \mu_0 = 0, 0 \leq \mu_1 \leq d_1 - 1, \ldots, 0 \leq \mu_{l-1} \leq d_{l-1} - 1, 1 \leq k \leq m \) in lexicographic order and number them from one to \( d_0 d_1 \cdots d_{l-1} m \). If a multi-index \((\mu_0, \mu_1, \ldots, \mu_{l-1}, j, k)\) is numbered by \( \alpha \), we write \( \alpha = (\mu_0, \mu_1, \ldots, \mu_{l-1}, k) \). With this notation, we denote by \( A(x_i, y) \) the square matrix of degree \( d_0 d_1 \cdots d_{l-1} m \) whose \((\alpha, \beta)\) component \( a_{\alpha \beta} \) is given by

\[
a_{\alpha \beta} = \delta_{\lambda_0 \mu_0} \cdots \delta_{\lambda_1 \mu_1} \cdots \delta_{\lambda_{l-1} \mu_{l-1}} \cdot \delta_{j_k, \lambda_{l-1}} \cdot x_{i_k} - \delta_{\lambda_0 \mu_0} \cdots \delta_{\lambda_{l-1} \mu_{l-1}} \cdot \delta_{j_k, \lambda_{l-1}} \cdot a_{\lambda_0 \lambda_1 \cdots \lambda_{l-1}, j_k}(y),
\]

where \( \alpha = (\mu_0, \mu_1, \ldots, \mu_{l-1}, j), \beta = (\mu_0, \mu_1, \ldots, \mu_{l-1}, k) \). Then the system of equations (A-1) is written as

\[
A(x_i, f(x)) \left( \begin{array}{c}
      f^* \left( \frac{\partial}{\partial y_1} \right) \\
      f^* \left( \frac{\partial}{\partial y_m} \right) \\
      x_1 f^* \left( \frac{\partial}{\partial y_1} \right) \\
      \vdots \\
      x_1 f^* \left( \frac{\partial}{\partial y_m} \right) \\
      x_0^{d_0-1} x_1^{d_1-1} \cdots x_{l-1}^{d_{l-1}-1} f^* \left( \frac{\partial}{\partial y_1} \right) \\
      \vdots \\
      x_0^{d_0-1} x_1^{d_1-1} \cdots x_{l-1}^{d_{l-1}-1} f^* \left( \frac{\partial}{\partial y_m} \right)
\end{array} \right) = \left( \begin{array}{c}
      tf(\xi_{0, 0, 0}) \\
      \vdots \\
      tf(\xi_{10, 0, 0}) \\
      \vdots \\
      tf(\xi_{10, 0, 0}) \\
      tf(\xi_{d_0-1, d_1-1, \ldots, d_{l-1}-1, 1}) \\
      \vdots \\
      tf(\xi_{d_0-1, d_1-1, \ldots, d_{l-1}-1, m})
\end{array} \right)
\]

We set \( \tilde{P}_i(x_i, y) = \det A(x_i, y) \), which is a monic polynomial of degree \( d_0 d_1 \cdots d_{l-1} m \) in \( x_i \) with coefficients in \( \mathcal{O}_C^{m, n} \). There exists a matrix \( \tilde{A}(x_i, y) \) such that

\[
\tilde{A}(x_i, y) \cdot A(x_i, y) = \tilde{P}_i(x_i, y) \cdot E_N,
\]

where \( N = d_0 d_1 \cdots d_{l-1} m \) and \( E_N \) denotes the identity matrix of degree \( N \). Operating \( \tilde{A}(x_i, f(x)) \) to both sides of (A-2), we have

\[
\tilde{P}_i(x_i, f(x)) \left( \begin{array}{c}
      f^* \left( \frac{\partial}{\partial y_1} \right) \\
      f^* \left( \frac{\partial}{\partial y_m} \right) \\
      x_1 f^* \left( \frac{\partial}{\partial y_1} \right) \\
      \vdots \\
      x_1 f^* \left( \frac{\partial}{\partial y_m} \right)
\end{array} \right) = \tilde{A}(x_i, f(x)) \left( \begin{array}{c}
      tf(\xi_{0, 0, 0}) \\
      tf(\xi_{0, 0, 0}) \\
      \vdots \\
      \vdots \\
      \vdots \\
      \vdots
\end{array} \right)
\]
This means
\[(A-3) \quad \hat{P}(x_i, f(x)) \cdot x_0 \phi x_1 \cdots x_{l_1 - 1} f^* \left( \frac{\partial}{\partial y_j} \right) \in tf(\Theta_{C^n, 0}).\]

We write
\[\hat{P}(x_i, y) = x_0 \phi + \hat{a}_n(y) x_1^{\hat{a}_1} + \cdots + \hat{a}_n(y),\]
where \(\hat{a}_n(y) \in \Theta_{C^n, 0}\) for \(1 \leq \beta \leq N\). If \(\hat{a}_n(0) = \cdots = \hat{a}_n(0) = 0\), then \(\hat{P}(x_i, y)\) is a Weierstrass polynomial in \(x_i\) with coefficients in \(\Theta_{C^n, 0}\). Otherwise, we denote by \(d_i\) the smallest \(\beta\) such that \(\hat{a}_n-\beta(0) \neq 0\). Then \(d_1 \geq 1\) by the assumption \(tf(\Theta_{C^n, 0}) = f^* \Theta_{C^n, 0}\), and \(d_i \geq 1\) for \(i \geq 2\) by the definition of \(\hat{P}(x_i, y)\). Hence we can write
\[\hat{P}(x_i, 0) = x_i^{d_i} (x_i^{d_i-1} + \cdots + \hat{a}_n(0));\]
that is, \(\hat{P}(x_i, y)\) is regular of order \(d_i\) with respect to \(x_i\); so by Weierstrass’ Preparation Theorem, \(\hat{P}(x_i, y)\) can be written as
\[\hat{P}(x_i, y) = u(x_i, y) \cdot P_i(x_i, y),\]
where \(u(x_i, y)\) is a unit of \(\Theta_{C^n, 0}\) and \(P_i(x_i, y)\) is a Weierstrass polynomial in \(x_i\) of degree \(d_i\). Therefore, by (A-3) we have
\[\frac{1}{u(x_i, f(x))} \hat{P}(x_i, f(x)) \cdot x_0 \phi x_1 \cdots x_{l_1 - 1} f^* \left( \frac{\partial}{\partial y_j} \right) \in tf(\Theta_{C^n, 0}).\]

This completes the proof of Lemma (A).

**Proposition (A):** Let \(f: (C^n, 0) \to (C^n, 0)\) be an infinitesimally stable germ of a holomorphic map. Then there exist closed polydiscs \(\bar{A}(n; r)\) and \(\bar{A}(m; r')\) in \(C^n\) and \(C^m\) respectively, and a representative \(\bar{f}\) of \(f\) defined in an open neighborhood of \(\bar{A}(n; r)\) which have the following properties:

(a) \(\bar{f}(\bar{A}(n; r)) \subset \bar{A}(n; r')\);

(b) for any \(\theta \in f^* \Theta_{C^n}(\bar{A}(n; r))\), there exist \(\xi \in \Theta_{C^n}(\bar{A}(n; r))\) and \(\zeta \in \Theta_{C^n}(\bar{A}(m; r'))\) such that \(\theta = \bar{f}^*(\xi) + \omega \bar{f}^*(\zeta)\) on \(\bar{A}(n; r)\);

(y) furthermore, the above \(\xi\) and \(\zeta\) satisfy the inequalities
\[\|\xi\|_{\bar{A}(n;r)} \leq K \|\theta\|_{\bar{A}(n;r)},\]
and
\[\|\zeta\|_{\bar{A}(m;r')} \leq K \|\theta\|_{\bar{A}(m;r')},\]
where \(K\) is a constant which depends only on \(f\) and does not on \(\theta\).

**Proof:** First we shall prove the proposition for the case where \(tf(\Theta_{C^n, 0}) \neq f^* \Theta_{C^m, 0}\).
By Lemma (A) there exist Weierstrass polynomials \(P_i(x_i, y)\) \((1 \leq i \leq n)\) and for every \(x_0 \phi x_1 \cdots x_{l_1 - 1} f^* \left( \frac{\partial}{\partial y_j} \right) \left( \lambda_0 = 0, 0 \leq \lambda_1 \leq d_1 - 1, \cdots, 0 \leq \lambda_{l_1 - 1} \leq d_{l_1 - 1} - 1, 1 \leq i \leq n, 1 \leq j \leq m \right)\), there exist \(\xi_{\lambda_0 \lambda_1 \cdots \lambda_{l_1 - 1}, j} \in \Theta_{C^n, 0}\) such that
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\[ P(x_i, f(x)) \cdot x_i^{\lambda_0} x_{i+1}^{\lambda_1} \cdots x_{i+m}^{\lambda_m} f^* \left( \frac{\partial}{\partial y_j} \right) = tf(\xi_{\lambda_0 \lambda_1 \cdots \lambda_{m-1}, j}), \]

and by the assumption that \( f \) is infinitesimally stable at 0, for every \( x_i^{\lambda_1} \cdots x_{i+m}^{\lambda_m} f^* \left( \frac{\partial}{\partial y_j} \right) \)
\( (0 \leq \lambda_1 \leq d_1 - 1, \ldots, 0 \leq \lambda_m \leq d_m - 1, 1 \leq j \leq m) \), there exist \( \xi_{\lambda_1 \cdots \lambda_m} \in \Theta_{C^n, 0} \) and \( \zeta_{\lambda_1 \cdots \lambda_m} \in \Theta_{C^m, 0} \) such that

\[ x_i^{\lambda_1} \cdots x_{i+m}^{\lambda_m} f^* \left( \frac{\partial}{\partial y_j} \right) = tf(\xi_{\lambda_1 \cdots \lambda_m}) + \omega f(\zeta_{\lambda_1 \cdots \lambda_m}). \]

We choose such \( \xi_{\lambda_0 \lambda_1 \cdots \lambda_{m-1}, j}, \zeta_{\lambda_1 \cdots \lambda_m}, \zeta_{\lambda_1 \cdots \lambda_m} \) and fix in the following.

We choose closed polydiscs \( \overline{A}(n; r) \) and \( \overline{A}(m; r') \) in \( C^n \) and \( C^m \), respectively, having the following properties:

(i) \( f \) has a representative \( \tilde{f} \) defined on an open neighborhood of \( \overline{A}(n; r) \);
(ii) \( \tilde{f}(\overline{A}(n; r)) \subseteq \overline{A}(m; r') \);
(iii) every Weierstrass polynomial \( P(x_i, y_1, \ldots, y_m) \) (\( 1 \leq i \leq n \)) in Lemma (A) is defined on an open neighborhood of \( \overline{A}(1; r) \times \overline{A}(m; r') \);
(iv) every \( \xi_{\lambda_0 \cdots \lambda_{m-1}, j}, \xi_{\lambda_1 \cdots \lambda_m} \) and \( \xi_{\lambda_1 \cdots \lambda_m} \) (\( 1 \leq i \leq n, 1 \leq j \leq m, \lambda_0 = 0, 0 \leq \lambda_1 \leq d_1 - 1, \ldots, 0 \leq \lambda_m \leq d_m - 1 \)) are defined in open neighborhoods of \( \overline{A}(n; r) \) and \( \overline{A}(m; r') \), respectively;
(v) for any \((b_1, \ldots, b_m) \in \overline{A}(m; r')\), all solutions of \( P(x_i, b_1, \ldots, b_m) = 0 \) satisfy \( |x_i| \leq r \) for \( 1 \leq i \leq n \).

By the property (v) above, we can apply Extended Weierstrass Division Theorem ([7], Chapter II, Sec. D) for each Weierstrass polynomial \( P(x_i, y_1, \ldots, y_m) \) (\( 1 \leq i \leq n \)) to the effect that for any \( g(x_i, \ldots, x_m, y_1, \ldots, y_m) \in \Theta(\overline{A}(n-i+1; r) \times \overline{A}(m; r')) \), there exist uniquely \( h(x_i, \ldots, x_m, y_1, \ldots, y_m) \in \Theta(\overline{A}(n-i+1; r) \times \overline{A}(m; r')) \) and \( q_\lambda(x_i, \ldots, x_m, y_1, \ldots, y_m) \in \Theta(\overline{A}(n-i; r) \times \overline{A}(m; r')) \) (\( 0 \leq \lambda \leq d_1 - 1 \)) such that

\[ g(x_i, \ldots, x_m, y_1, \ldots, y_m) = h(x_i, \ldots, x_m, y_1, \ldots, y_m) P(x_i, y_1, \ldots, y_m) + \sum_{\lambda=0}^{d_1-1} q_\lambda(x_{i+1}, \ldots, x_m, y_1, \ldots, y_m) x_i^{\lambda} \]

and there exists a constant \( K_i \) which depends only on \( P(x_i, y) \) and does not on \( g(x_i, \ldots, x_m, y_1, \ldots, y_m) \) such that

\[ \| h \|_{\overline{A}(n-i+1; r) \times \overline{A}(m; r')} \leq K_i \| g \|_{\overline{A}(n-i+1; r) \times \overline{A}(m; r')}, \quad \text{and} \]
\[ \| q_\lambda \|_{\overline{A}(n-i; r) \times \overline{A}(m; r')} \leq K_i \| g \|_{\overline{A}(n-i+1; r) \times \overline{A}(m; r')} \]

for \( 0 \leq \lambda \leq d_1 - 1 \).

Let \( \theta = \sum_{j=1}^{m} a_j(x) f^* \left( \frac{\partial}{\partial y_j} \right) \in f^* \Theta_{C^m}(\overline{A}(n; r)) \) be given. We identify \( \theta \) with

\[ \tilde{\theta} = \sum_{j=1}^{m} a_j(x) \left( \frac{\partial}{\partial y_j} \right)_{(x, f(x))} \in \Gamma(G_f, \pi_{C^m}^* \Theta_{C^m} | G_f), \]

i.e., a cross-section of the sheaf \( \pi_{C^m}^* \Theta_{C^m} | G_f \) over \( G_f \), where \( G_f \) denotes the graph of \( f \) in \( C^n \times C^m \) and \( \pi_{C^m} \) the natural projection: \( C^n \times C^m \to C^m \). We regard \( \tilde{\theta} \) as an element of \( \Theta_{C^n \times C^m}(\overline{A}(n; r) \times \overline{A}(m; r')) \).

We claim that:

1. for each \( i = 1, \ldots, n \), \( \tilde{\theta} \) can be expressed as
\begin{align*}
\theta &= \sum_{j=1}^{m} b_j(x, y)P_1(x_1, y)(\frac{\partial}{\partial y_j}(x, y)) \\
&\quad + \sum_{j=1}^{m} \sum_{\lambda_j=1}^{d_j-1} b_{\lambda_j}(x_2, \ldots, x_n, y) P_2(x_2, y) x_0^\lambda x_1^j (\frac{\partial}{\partial y_j}(x, y)) \\
&\quad + \sum_{j=1}^{m} \sum_{\lambda_j=1}^{d_j-1} c_{\lambda_j}(x_2, \ldots, x_n, y) x_0^\lambda x_1^j (\frac{\partial}{\partial y_j}(x, y)),
\end{align*}
(A-4)

\begin{align*}
&\text{where } b_j(x, y) \in \mathcal{O}_{C^n \times C^m}(\overline{A}(n; r) \times \overline{A}(m; r)), \ b_{\lambda_j}(x_2, \ldots, x_n, y) \in \mathcal{O}_{C^n \times C^m}(\overline{A}(n-i+1; r) \times \overline{A}(m; r)), \ c_{\lambda_j}(x_2, \ldots, x_n, y) \in \mathcal{O}_{C^n \times C^m}(\overline{A}(n-i; r) \times \overline{A}(m; r)) \text{ for } 1 \leq j \leq m, 1 \leq \lambda_j \leq d_j - 1, 1 \leq i' \leq i; \\
&\quad \text{(2)} \quad \|b_j(x, y)\|_{\overline{A}(m; r) \times \overline{A}(n; r)} \leq K_1 \|a_j(x)\|_{\overline{A}(m; r)}, \quad \text{and} \\
&\quad \|c_{\lambda_j}(x_2, \ldots, x_n, y)\|_{\overline{A}(n-i; r) \times \overline{A}(m; r)} \leq K_1 \times \cdots \times K_r \|a_j(x)\|_{\overline{A}(m; r)} \text{ for } 1 \leq j \leq m, 1 \leq \lambda_j \leq d_j - 1, 1 \leq i' \leq i. \\
\end{align*}
(A-5)

Assume that the above claim is proved for all natural number < i, and we shall prove it for i \geq 1. For any c_{\lambda_j}(x_2, \ldots, x_n, y) in the expression at (A-4)_{i-1}, by Extended Weierstrass Division Theorem, there exist uniquely

\begin{align*}
&\quad \text{such that:} \\
&\quad \text{(1)} \quad c_{\lambda_j}(x_2, \ldots, x_n, y) \\
&\quad \text{(2)} \quad \|c_{\lambda_j}(x_2, \ldots, x_n, y)\|_{\overline{A}(n-i+1; r) \times \overline{A}(m; r)} \leq K_1 \times \cdots \times K_r \|a_j(x)\|_{\overline{A}(m; r)} \\
&\quad \text{for } 0 \leq \lambda_j \leq d_j - 1, \\
\end{align*}
(A-6)

\begin{align*}
&\quad \text{Substituting (A-6) into (A-4)_{i-1}, we have (A-4)_i. By (A-5)_{i-1} and (A-7), we have} \\
&\quad \text{for } 0 \leq \lambda_j \leq d_j - 1; \text{ hence we have (A-5)_i. Therefore the claim is proved.} \\
&\quad \text{Restricting the equality (A-4)_n to the graph } G_f \text{ of } f \text{ in } C^n \times C^m, \text{ we have} \\
\theta &= \sum_{j=1}^{m} b_j(x, f(x)) P_j(x_1, f(x)) f^* (\frac{\partial}{\partial y_j},)
\end{align*}
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\[ + \sum_{j=1}^{d_1} \sum_{\lambda_j = 1}^{d_1-1} b_{\lambda_j, f(x_2, \ldots, x_n, f(x))} P_1(x_2, f(x)) x_1^{d_1} x_1^{d_2} f^* \left( \frac{\partial}{\partial y_j} \right)_x \]

\[ + \sum_{j=1}^{d_1} \sum_{\lambda_j = 1}^{d_1-1} \cdots \sum_{\lambda_{n-1} = 1}^{d_1-1} b_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} P_{n-1}(x_2, f(x)) x_1^{d_1} x_1^{d_2} \cdots x_n^{d_1-1} f^* \left( \frac{\partial}{\partial y_j} \right)_x \]

\[ + \sum_{j=1}^{d_1} \sum_{\lambda_j = 1}^{d_1-1} \cdots \sum_{\lambda_{n-1} = 1}^{d_1-1} c_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} x_1^{d_1} x_1^{d_2} \cdots x_n^{d_1-1} f^* \left( \frac{\partial}{\partial y_j} \right)_x \]

(A-8)

Substituting

\[ P_{i'}(x_2, f(x)) x_1^{d_1} x_1^{d_2} \cdots x_n^{d_1-1} f^* \left( \frac{\partial}{\partial y_j} \right)_x = tf(\xi_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))}) \]

(\(1 \leq i' \leq n\), \(1 \leq j \leq m\)), and

\[ x_1^{d_1} x_1^{d_2} \cdots x_n^{d_1-1} f^* \left( \frac{\partial}{\partial y_j} \right)_x = tf(\xi_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))}) + \omega f(\xi_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))}) \]

(\(1 \leq j \leq m\)) into (A-8), we have

\[ \theta = \sum_{j=1}^{m} b_j(x_2, f(x)) f(\xi_{0,j}) \]

\[ + \sum_{j=1}^{m} \sum_{\lambda_j = 1}^{d_1-1} \cdots \sum_{\lambda_{n-1} = 1}^{d_1-1} b_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \xi_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \]

\[ + \sum_{j=1}^{m} \sum_{\lambda_j = 1}^{d_1-1} \cdots \sum_{\lambda_{n-1} = 1}^{d_1-1} c_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \xi_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \]

Hence, if we set

\[ \xi = \sum_{j=1}^{m} b_j(x_2, f(x)) \xi_{0,j} \]

\[ + \sum_{j=1}^{m} \sum_{\lambda_j = 1}^{d_1-1} \cdots \sum_{\lambda_{n-1} = 1}^{d_1-1} b_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \xi_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \]

\[ + \sum_{j=1}^{m} \sum_{\lambda_j = 1}^{d_1-1} \cdots \sum_{\lambda_{n-1} = 1}^{d_1-1} c_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \xi_{\lambda_1 \cdots \lambda_{n-1}, f(x_2, \ldots, x_n, f(x))} \]

then \( \xi \in \Theta_{C^n}(\tilde{\alpha}(n; r'), \xi) \in \Theta_{C^n}(\tilde{\alpha}(m; r')) \) and \( \theta = tf(\xi) + \omega f(\xi) \).

Furthermore, if we set

\[ K' = \text{Max} \{ K_1_i \} \]

\[ L = \text{Max} \{ \| \xi_{\lambda_0 \cdots \lambda_{n-1}} \|_{\tilde{\alpha}(m; r')} , \| \xi_{\lambda_0 \cdots \lambda_{n-1}} \|_{\tilde{\alpha}(m; r')} \} \]

\[ = \text{Max} \{ \| \xi_{\lambda_0 \cdots \lambda_{n-1}} \|_{\tilde{\alpha}(m; r')} , \| \xi_{\lambda_0 \cdots \lambda_{n-1}} \|_{\tilde{\alpha}(m; r')} \} \]
then by (A-5)\(_h\), we have
\[ \| \xi \|_{\mathfrak{A}(n,r')} \leq (K'mL + K'^2md_1L + \ldots + K'^mmd_1 \ldots d_nL)\| \theta \|_{\mathfrak{A}(n,r)}, \]
and
\[ \| \xi \|_{\mathfrak{A}(m,r')} \leq (K'md_1 \times \ldots \times d_nL)\| \theta \|_{\mathfrak{A}(n,r)}. \]
Therefore, if we set
\[ K = K'mL + K'^2md_1L + \ldots + K'^mmd_1 \ldots d_nL, \]
then \( K \) is a constant which depends only on \( f \) and does not on \( \theta \), and
\[ \| \xi \|_{\mathfrak{A}(n,r')} \leq K\| \theta \|_{\mathfrak{A}(n,r)}, \quad \| \xi \|_{\mathfrak{A}(m,r')} \leq K\| \theta \|_{\mathfrak{A}(n,r)} \]
holds. This completes the proof of the proposition for the case where \( tf(\Theta_{c^m,0}) \neq f^*\Theta_{c^m,0} \).

It is rather easy to show the proposition for the case where \( tf(\Theta_{c^m,0}) = f^*\Theta_{c^m,0} \).
We choose closed polydiscs \( \mathfrak{A}(n; r) \) and \( \mathfrak{A}(m; r') \) in \( C^n \) and \( C^m \), respectively, having the following property in addition to the ones (i), (ii) before.

(iii) for every \( f^* \left( \frac{\partial}{\partial y_j} \right) \in f^*\Theta_{c^m,0} (1 \leq j \leq m) \) there exist \( \xi_j \in \Theta_{c^m}(\mathfrak{A}(n; r)) \) such that \( tf(\xi_j) = f^* \left( \frac{\partial}{\partial y_j} \right) \).

Then, setting \( \xi = \sum_{j=1}^{m} a_j(x)\xi_j \) for any \( \theta = \sum_{j=1}^{m} a_j(x)f^* \left( \frac{\partial}{\partial y_j} \right) \in f^*\Theta_{c^m}(\mathfrak{A}(n; r)), \) we have \( \xi \in \Theta_{c^m}(\mathfrak{A}(n; r)) \) and \( tf(\xi) = \theta \). Furthermore, we have
\[ \| \xi \|_{\mathfrak{A}(n,r')} \leq \sum_{j=1}^{m} \| a_j(x)\|_{\mathfrak{A}(n,r')}\| \xi_j \|_{\mathfrak{A}(n,r')} \]
\[ \leq K\| \theta \|_{\mathfrak{A}(n,r')}, \]
where we set \( K = \text{Max} \| \xi_j \|_{\mathfrak{A}(n,r')} \) which depends only on \( f \) and does not on \( \theta \). Thus the proposition surely holds also in this case. Q.E.D.

3°) It is not so difficult to extend Proposition (A) to the case of a simultaneously infinitesimally stable multi-germ of a holomorphic map.

Proposition (B): Let \( S = \{ p_1, \ldots, p_s \} \) be a finite subset of \( C^n \) and \( f: (C^n, S) \rightarrow (C^m, 0) \) a simultaneously infinitesimally stable multi-germ of a holomorphic map. Then there exist closed polydiscs \( \mathfrak{A}(n; p_i; r_i) \) (\( 1 \leq i \leq s \)) in \( C^n \) which are mutually disjoint, a closed polydisc \( \mathfrak{A}(m; r') \) in \( C^m \), and a representative \( \hat{f} \) of \( f \) defined in an open neighborhood of \( \bigotimes_{i=1}^{s} \mathfrak{A}(n; p_i; r_i) \), the disjoint union of \( \mathfrak{A}(n; p_i; r_i) \) (\( 1 \leq i \leq s \)) which have the following properties:

(a) \( \hat{f}(\bigotimes_{i=1}^{s} \mathfrak{A}(n; p_i; r_i)) \subset \mathfrak{A}(m; r') \);

(b) for any \( \theta \in f^*\Theta_{c^m}(\bigotimes_{i=1}^{s} \mathfrak{A}(n; p_i; r_i)) \), there exist \( \xi \in \Theta_{c^m}(\bigotimes_{i=1}^{s} \mathfrak{A}(n; p_i; r_i)) \) and \( \zeta \in \Theta_{c^m}(\mathfrak{A}(m; r')) \) such that \( \theta = tf(\xi) + \omega f(\zeta) \) on \( \bigotimes_{i=1}^{s} \mathfrak{A}(n; p_i; r_i) \);

(c) furthermore, the above \( \xi \) and \( \zeta \) satisfy the inequalities
\[ \| \xi \|_{\mathfrak{A}(n; p_i; r_i)} \leq K\| \theta \|_{\mathfrak{A}(n; p_i; r_i)}, \]
\[ \| \zeta \|_{\mathfrak{A}(m; r')} \leq K\| \theta \|_{\mathfrak{A}(n; p_i; r_i)}, \]
where $K$ is a constant which depends only on $f$ and does not on $\theta$.

Proof: We set $f^{(i)}(\cdot) = f_{(\xi^{(i)} \cdot \theta)}(\mathbb{C}^n, p_i) \to (\mathbb{C}^n, 0)$ for $i = 1, \ldots, s$. Then by the assumption that $f$ is simultaneously infinitesimally stable at $S$, each $f^{(i)}(\cdot): (\mathbb{C}^n, p_i) \to (\mathbb{C}^m, 0)$ ($1 \leq i \leq s$) is infinitesimally stable at $p_i$. First we prove the proposition under the assumption that $t f^{(i)}(\Theta_{\mathbb{C}^n, p_i}) \neq f^{(i)}(\cdot) \Theta_{\mathbb{C}^n, p_i}$ for $1 \leq i \leq s$. By Lemma (A) there exist Weierstrass polynomials $P_{x_1}^{(i)}(x_{i_1}, y), P_{x_2}^{(i)}(x_{i_2}, y), \ldots, P_{x_n}^{(i)}(x_{i_n}, y)$ which have the properties stated in Lemma (A) for each $f^{(i)}(\cdot)$ ($1 \leq i \leq s$), where $(x_{i_1}, \ldots, x_{i_n}) = (x_1 - x_{i_1}(p_i), \ldots, x_n - x_{i_n}(p_i))$ denotes a local coordinate centered at $p_i$. We denote by $d_{x_1}^{(i)}$, the degree of $P_{x_1}^{(i)}(x_{i_1}, y)$ for $1 \leq i \leq s$, $1 \leq \alpha \leq n$. By Lemma (A), for every $x_{i_1}^{d_{x_1}^{(i)}} \cdots x_{i_n}^{d_{x_n}^{(i)}} f^{(i)*} \left( \frac{\partial}{\partial y_\beta} \right)$, $(\lambda_0 = 0, 0 \leq \lambda_1 \leq d_{x_1}^{(i)} - 1, \ldots, 0 \leq \lambda_{n-1} \leq d_{x_n}^{(i)} - 1, 1 \leq \alpha \leq n, 1 \leq \beta \leq m, 1 \leq i \leq s)$, there exists $\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta} \in \Theta_{\mathbb{C}^n, p_i}$ such that

(A-9) \[ P_{x_1}^{(i)}(x_{i_1}, f^{(i)}(x_{i_1})) \cdots x_{i_1}^{d_{x_1}^{(i)}} \cdots x_{i_n}^{d_{x_n}^{(i)}} f^{(i)*} \left( \frac{\partial}{\partial y_\beta} \right) = t f^{(i)}(\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta}), \]

and since $f$ is infinitesimally stable at $S$, for every $x_{i_1}^{\lambda_{i_1}} \cdots x_{i_n}^{\lambda_{i_n}} f^{(i)}* \left( \frac{\partial}{\partial y_\beta} \right)$ there exist $\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta} \in \Theta_{\mathbb{C}^n, S}$ and $\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta} \in \Theta_{\mathbb{C}^n, 0}$ such that

(A-10) \[ \left( 0, \ldots, 0, x_{i_1}^{d_{x_1}^{(i)}} \cdots x_{i_n}^{d_{x_n}^{(i)}} f^{(i)*} \left( \frac{\partial}{\partial y_\beta} \right), 0, \ldots, 0 \right) \in f^{(i)}(\Theta_{\mathbb{C}^n, S}) \]

= $t f^{(i)}(\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta}) + \omega f^{(i)}(\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta})$.

We choose such $\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta}, \xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta}, \xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta}$ and fix in the following. We choose closed polydiscs $\tilde{D}(n; p_i; r_i) (1 \leq i \leq s)$ and $\tilde{D}(m; r)$ in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, having the following properties:

(i) $f$ has a representative $\tilde{f}$ defined in an open neighborhood of $\bigoplus_{i=1}^s \tilde{D}(n; p_i; r_i)$;

(ii) $f(\bigoplus_{i=1}^s \tilde{D}(n; p_i; r_i)) \subset \tilde{D}(m; r')$;

(iii) every Weierstrass polynomials $P_{x_1}^{(i)}(x_{i_1}, y) (1 \leq \alpha \leq n, 1 \leq i \leq s)$ is defined on an open neighborhood of $\tilde{D}(1; r) \times \tilde{D}(m; r')$;

(iv) every $\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta}$ $(\lambda_0 = 0, 0 \leq \lambda_1 \leq d_{x_1}^{(i)} - 1, \ldots, 0 \leq \lambda_{n-1} \leq d_{x_n}^{(i)} - 1, 1 \leq \alpha \leq n, 1 \leq \beta \leq m, 1 \leq i \leq s)$ is defined on an open neighborhood of $\tilde{D}(n; p_i; r_i)$ and $\xi^{(i)}_{\lambda_1, \ldots, \lambda_{n-1}, \beta}$, $(0 \leq \lambda_1 \leq d_{x_1}^{(i)} - 1, \ldots, 0 \leq \lambda_{n-1} \leq d_{x_n}^{(i)} - 1, 1 \leq \beta \leq m, 1 \leq i \leq s)$, are defined on an open neighborhood of $\bigoplus_{i=1}^s \tilde{D}(n; p_i; r_i)$ and $\tilde{D}(m; r')$ respectively;

(v) for any $(b_1, \ldots, b_m) \in \tilde{D}(m; r')$ all solutions of $P_{x_1}^{(i)}(x_{i_1}, b_1, \ldots, b_m) = 0$ satisfy $|x_{i_1}| \leq r_i$ for $1 \leq \alpha \leq n, 1 \leq i \leq s$.

Let $\theta = (\theta_1, \ldots, \theta_s) \in f^* \Theta_{\mathbb{C}^m, S} = f^* \Theta_{\mathbb{C}^m, p_1} \times \cdots \times f^* \Theta_{\mathbb{C}^m, p_s}$ be given, then $\theta$ can be written as

$\theta = (\theta_1, 0, \ldots, 0) + (0, \theta_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, \theta_s),$

where $\theta_i \in f^* \Theta_{\mathbb{C}^m, p_i} (1 \leq i \leq s)$. Hence it suffices to prove the proposition for $\theta$ in the form $(0, \ldots, 0, \theta_i, 0, \ldots, 0), (0, \theta_i, 0, \ldots, 0)$, $\theta_i \in f^* \Theta_{\mathbb{C}^m, p_i}$. Let such $\theta = (0, \ldots, 0, \theta_i, 0, \ldots, 0)$ be given, then by the same arguments as in the proof of Lemma (A), $\theta_i$ can be written as

$\theta_i = \sum_{\beta=1}^{d_{x_1}^{(i)}} \frac{\partial}{\partial y_\beta} (x_{i_1} f^{(i)}(x_{i_1})) P_{x_1}^{(i)}(x_{i_1}, f^{(i)}(x_{i_1})) f^{(i)*} \left( \frac{\partial}{\partial y_\beta} \right)$

$+$

$\cdots$
(A-11)

\[ + \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} c_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)}(x_{i_1}, f^{(i)}(x_i)) P_{n}^{(i)}(x_{i_1}, f^{(i)}(x_i)) \]
\[ \times x_{\lambda_1}^{(i)} x_{\lambda_{n-1}}^{(i)} \cdots x_{\lambda_n}^{(i)} f^{(i)} \left( \frac{\partial}{\partial y_{\beta_{i_1}}} \right) \]
\[ + \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} \sum_{\lambda_n=0}^{d_{i_n}^{(1)}-1} c_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)}(x_{i_1}, f^{(i)}(x_i)) x_{\lambda_1}^{(i)} x_{\lambda_{n-1}}^{(i)} x_{\lambda_n}^{(i)} f^{(i)} \left( \frac{\partial}{\partial y_{\beta_{i_1}}} \right) \]
\[ \text{where } b_{\beta}^{(i)}(x_i, y) \in \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^m}(\mathcal{A}(n; p_i; r_i) \times \mathcal{A}(m; r')), b_{\beta}^{(i)}(x_{i_1}, \ldots, x_{i_n}, y) \in \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}^m}(\mathcal{A}(n; \alpha+1; p_i; r_i) \times \mathcal{A}(m; r')), c_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)}(y) \in \mathcal{O}_{\mathbb{C}^n}(\mathcal{A}(m; r')); \]
\[ \text{for } 1 \leq \beta \leq m, 1 \leq \lambda_1 \leq d_{i_1}^{(1)}-1, \ldots, 1 \leq \lambda_n \leq d_{i_n}^{(1)}-1, 1 \leq \alpha \leq n, 1 \leq i \leq s. \]

Then by (A-9) and (A-10), we derive from (A-11) that

\[ \theta = (0, \ldots, 0, \theta_{i_1}, 0, \ldots, 0) \]
\[ = tf((0, \ldots, 0, \sum_{\beta_1=1}^{m} b_{\beta}^{(i)}(x_i, f^{(i)}(x_i)) \xi_{\beta_{i_1}}^{(1)}(x_i), 0, \ldots, 0)) \]
\[ + \cdots \]
\[ + \cdots \]
\[ + \cdots \]
\[ + tf((0, \ldots, 0, \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} b_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)}(x_{i_1}, f^{(i)}(x_i)) \xi_{\lambda_1,\ldots,\lambda_n,\beta}^{(1)}(x_i), 0, \ldots, 0)) \]
\[ + tf((0, \ldots, 0, \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} \sum_{\lambda_n=0}^{d_{i_n}^{(1)}-1} f^*(c_{\lambda_1,\ldots,\lambda_n,\beta}(y)) \xi_{\lambda_1,\ldots,\lambda_n,\beta}^{(i))} \]
\[ + \omega f((0, \ldots, 0, \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} \sum_{\lambda_n=0}^{d_{i_n}^{(1)}-1} c_{\lambda_1,\ldots,\lambda_n,\beta}(y) \xi_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)})) \]

where \( f^*(c_{\lambda_1,\ldots,\lambda_n,\beta}(y)) \in \mathcal{O}_{\mathbb{C}^n, S} \) denotes the pull-back of \( c_{\lambda_1,\ldots,\lambda_n,\beta}(y) \in \mathcal{O}_{\mathbb{C}^n, 0} \) by the canonical homomorphism \( f^*: \mathcal{O}_{\mathbb{C}, 0} \rightarrow \mathcal{O}_{\mathbb{C}, S} \) and \( f^*(c_{\lambda_1,\ldots,\lambda_n,\beta}(y)) \xi_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)} \) stands for multiplying \( \xi_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)} \) by \( f^*(c_{\lambda_1,\ldots,\lambda_n,\beta}(y)) \) with respect to the \( \mathcal{O}_{\mathbb{C}, S} \)-module structure of \( \mathcal{O}_{\mathbb{C}, S} \). Hence if we set

\[ \xi = (0, \ldots, 0, \sum_{\beta_1=1}^{m} b_{\beta}^{(i)}(x_i, f^{(i)}(x_i)) \xi_{\beta_{i_1}}^{(1)}(x_i), 0, \ldots, 0) \]
\[ + \cdots \]
\[ + \cdots \]
\[ + \cdots \]
\[ + (0, \ldots, 0, \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} b_{\lambda_1,\ldots,\lambda_n,\beta}(x_{i_1}, f^{(i)}(x_i)) \xi_{\lambda_1,\ldots,\lambda_n,\beta}(x_i), 0, \ldots, 0) \]
\[ + \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} f^*(c_{\lambda_1,\ldots,\lambda_n,\beta}(y)) \xi_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)}; \text{ and} \]
\[ \xi = \sum_{\beta_1=1}^{m} \sum_{\lambda_1=0}^{d_{i_1}^{(1)}-1} \sum_{\lambda_{n-1}=0}^{d_{i_{n-1}}^{(1)}-1} \sum_{\lambda_n=0}^{d_{i_n}^{(1)}-1} c_{\lambda_1,\ldots,\lambda_n,\beta}(y) \xi_{\lambda_1,\ldots,\lambda_n,\beta}^{(i)}, \]
then $\xi \in \Theta_{C^0}\left(\bigoplus_{i=1}^{s} \tilde{A}(n; p_i; r_i)\right)$, $\zeta \in \Theta_{C^0}\left(\tilde{A}(m; r')\right)$ and $\theta = t f(\xi) + w f(\zeta)$ holds on $\bigoplus_{i=1}^{s} \tilde{A}(n; p_i; r_i)$.

Furthermore, by the almost identical arguments as in the proof of Proposition (A), we can prove that there exists a constant $K$ which depends only on $f$, and does not on $\theta$ such that

$$\|\xi\|_{\tilde{A}(n; p_i; r_i)} \leq K \|\theta\|_{\tilde{A}(n; p_i; r_i)}, \quad \text{and}$$

$$\|\zeta\|_{\tilde{A}(m; r')} \leq K \|\theta\|_{\tilde{A}(m; p_i; r_i)}.$$

This completes the proof of the proposition for the case where $t f^{(i)}(\theta_{C^0}, p_i) \neq f^{(i)}(\theta_{C^0}, p_i)$ for $1 \leq i \leq s$.

Next we shall prove the proposition for general case. We may assume that $t f^{(i)}(\theta_{C^0}, p_i) = f^{(i)}(\theta_{C^0}, p_i)$ for $1 \leq i \leq s < s'$, and $t f^{(i)}(\theta_{C^0}, p_i) = f^{(i)}(\theta_{C^0}, p_i)$ for $s' + 1 \leq i \leq s$. We set $S' := \{p_1, \ldots, p_{s'}\}$ and $f' := f|_{C^0, S'}: (C^0, S') \rightarrow (C^0, 0)$. Then the proposition holds for $f'$. Hence there exists a disjoint sum $\bigoplus_{i=1}^{s'} \tilde{A}(n; p_i; r_i)$ of closed polydiscs in $C^0$, a polydisc $\tilde{A}(m; r')$ in $C^0$, a representative $f'$ of $f'$ defined on an open neighborhood of $\bigoplus_{i=1}^{s'} \tilde{A}(n; p_i; r_i)$, and a constant $K'$ which have the properties (a), (b), (y) stated in the proposition. For each $i$ with $s' + 1 \leq i \leq s$, there exists a closed polydisc $\tilde{A}(n; p_i; r_i)$ in $C^0$, and a representative $\tilde{f}^{(i)}$ of $f^{(i)}$ defined in an open neighborhood of $\tilde{A}(n; p_i; r_i)$, which have the following properties:

(i) $f^{(i)}(\tilde{A}(n; p_i; r_i)) = \tilde{A}(m; r')$;

(ii) for any $\theta_i \in (f^{(i)*})\Theta_{C^0}(\tilde{A}(n; p_i; r_i))$ there exists $\xi_i \in \Theta_{C^0}(\tilde{A}(n; p_i; r_i))$ such that $t f^{(i)}(\xi_i) = \theta_i$;

(iii) furthermore, this $\xi_i$ satisfies

$$\|\xi_i\|_{\tilde{A}(n; p_i; r_i)} \leq K_i \|\theta_i\|_{\tilde{A}(n; p_i; r_i)},$$

where $K_i$ is a constant which depends only on $f^{(i)}$ and does not on $\theta_i$.

Let $\theta = (\theta_1, \ldots, \theta_s) \in f^* \Theta_{C^0}\left(\bigoplus_{i=1}^{s'} \tilde{A}(n; p_i; r_i)\right)$ be given. We set $\theta' = (\theta_1, \ldots, \theta_s) \in f^* \Theta_{C^0}\left(\bigoplus_{i=1}^{s'} \tilde{A}(n; p_i; r_i)\right)$. Then there exist $\xi' = (\xi_1, \ldots, \xi_{s'}) \in \Theta_{C^0}\left(\bigoplus_{i=1}^{s'} \tilde{A}(n; p_i; r_i)\right)$ and $\zeta \in \Theta_{C^0}(\tilde{A}(m; r'))$ such that

$$\theta' = t f'(\xi') + w f'(\zeta),$$

$$\|\xi'|\|_{\tilde{A}(n; p_i; r_i)} \leq K' \|\theta'\|_{\tilde{A}(n; p_i; r_i)}; \quad \text{and}$$

$$\|\zeta\|_{\tilde{A}(m; r')} \leq K' \|\theta'\|_{\tilde{A}(m; p_i; r_i)}.$$

On the other hand, for each $\theta^{(i)} + w f'(\xi)$ with $s' + 1 \leq i \leq s$, there exists $\xi_i \in \Theta_{C^0}(\tilde{A}(n; p_i; r_i))$ such that

$$t f^{(i)}(\xi_i) = \theta_i + w f^{(i)}(\xi)$$

and

$$\|\xi_i\|_{\tilde{A}(n; p_i; r_i)} \leq K_i \|\theta_i\|_{\tilde{A}(n; p_i; r_i)} + K_i \|\xi\|_{A^0(\theta_i, r_i)}$$

$$\leq K_i \|\theta_i\|_{\tilde{A}(n; p_i; r_i)} + K_i \|\xi\|_{\tilde{A}(m; r')}.$$
\[ \leq K' \| \theta \|_F \hat{a}(n; p; r; \xi) + K' \| \theta' \|_F \hat{a}(n; p; r; \xi) \]
\[ \leq K'' \| \theta \|_F \hat{a}(n; p; r; \xi), \]

where we set \( K'' = \max \{ K(1 + K') \} \) which depends only on \( f \) and does not on \( \theta \).

Therefore, if we set
\[ \xi = (\xi_1, \ldots, \xi_s', \xi_{s'}+1, \ldots, \xi_s) \in \Theta_{C^n} \left( \bigoplus_{i=1}^s \hat{a}(n; p_i; r_i) \right) \]
and \( K = \max \{ K', K'' \} \), we have
\[ f(\xi) + \omega f(\xi) = \theta; \]
\[ \| \xi \|_F \hat{a}(n; p; r; \xi) \leq K \| \theta \|_F \hat{a}(n; p; r; \xi); \]
and
\[ \| \xi \|_F \hat{a}(m; r') \leq K \| \theta \|_F \hat{a}(n; p; r). \]

This completes the proof of the proposition.

4°) Theorem (A): Let \( S = \{ p_1, \ldots, p_s \} \) be a finite subset of \( C^n \), and \( f: (C^n, S) \to (C^n, 0) \) a multi-germ of a holomorphic map. Then \( f \) is simultaneously stable at \( S \) if, and only if, it is simultaneously infinitesimally stable at \( S \).

Proof: The proof of 'only if' part of this theorem is easy, so we omit it. Here we shall show the proof of 'if' part of this theorem; i.e., if a multi-germ \( f: (C^n, S) \to (C^n, 0) \) of a holomorphic map is simultaneously infinitesimally stable, then for any unfolding \( F: (C^n \times C', S \times 0) \to (C^n \times C', 0 \times 0) \) of the multi-germ \( F: (C^n, S) \to (C^n, 0) \), there exist germs of \( t \)-levels \( (t \in C') \) preserving analytic automorphisms \( G: (C^n \times C', S \times 0) \to (C^n \times C', S \times 0) \) and \( H: (C^n \times C', 0 \times 0) \to (C^n \times C', 0 \times 0) \) with \( G_{|_{(C^n \times 0, S \times 0)}} = id_{C^n} \) and \( H_{|_{(C^n \times 0, 0 \times 0)}} = id_{C^n} \), such that the following diagram

\[ \begin{array}{ccc}
(C^n \times C', S \times 0) & \xrightarrow{G} & (C^n \times C', S \times 0) \\
F \times id_{C'} & \downarrow \phi & \\
(C^n \times C', 0 \times 0) & \xrightarrow{H} & (C^n \times C', 0 \times 0)
\end{array} \]

is commutative.

We write \( \tilde{F}(x, t) = (F(x, t), t), \tilde{G}(x, t) = (G(x, t), t), \tilde{H}(x, t) = (H(x, t), t) \). With these notations the commutativity of the diagram (A-12) is equivalent to

\[ F(G(x, t), t) = H(F(x, t), t). \]

The conditions \( \tilde{G}_{|_{(C^n \times 0, S \times 0)}} = id_{C^n} \) and \( \tilde{H}_{|_{(C^n \times 0, 0 \times 0)}} = id_{C^n} \) imply that

\[ G(x, 0) = x \quad \text{and} \quad H(y, 0) = y. \]

We expand \( F(x, t), G(x, t) \) and \( H(y, t) \) as power series in \( t \) with coefficients in vector-valued holomorphic functions in \( x \) and \( y \), respectively, which we write as

\[ F(x, t) = \sum_{k \geq 0} F_k(x, t), \]
\[ G(x, t) = \sum_{k \geq 0} G_k(x, t), \]
\[ H(y, t) = \sum_{k \geq 0} H_k(y, t), \]

where \( F_k(x, t) \) (resp. \( G_k(x, t) \)) is a homogeneous polynomial in \( (t_1, \ldots, t_l) \) of degree \( k \).
with coefficients in \( m \)-vector valued (resp. \( n \)-vector valued) holomorphic functions in \( x \), and \( H_\mu(x, t) \) the one with coefficients in \( m \)-vector valued holomorphic functions in \( y \). We set

\[
F^\mu(x, t) = \sum_{0 \leq k \leq \mu} F_k(x, t),
\]

\[
G^\mu(x, t) = \sum_{0 \leq k \leq \mu} G_k(x, t),
\]

\[
H^\mu(y, t) = \sum_{0 \leq k \leq \mu} H_k(y, t).
\]

In the following we identify a holomorphic function with its power series expansion.

1) \textit{Existence of formal solutions:}

First we prove the existence of formal power series \( G(x, t) \) and \( H(y, t) \) satisfying (A-13) and (A-14). In the subsequence, for any power series \( P(t), Q(t) \) in \( (t_1, \ldots, t_i) \) with coefficients in a certain module, we indicate by writing \( P(t) \equiv Q(t) \) that the power series expansion of \( P(t) - Q(t) \) in \( (t_1, \ldots, t_i) \) contains no terms of degree \( \leq \mu \). With this notation (A-13) is equivalent to the equations of congruences

(A-15)_\mu \quad F(G^\mu(x, t), t) \equiv H^\mu(f(x), t) \quad (\mu = 0, 1, 2, \ldots)

Here we consider \( F(G^\mu(x, t), t) \) and \( H^\mu(f(x), t) \) are power series in \( (t_1, \ldots, t_i) \) with coefficients in \( m \)-vector valued holomorphic functions in \( (x_1, \ldots, x_n) \). We construct \( G^\mu(x, t) \) and \( H^\mu(y, t) \) satisfying (A-15)_\mu and (A-14) by induction on \( \mu \).

For \( \mu = 0 \), by (A-14) we have \( G^0(x, t) = x \) and \( H^0(y, t) = y \). The condition (A-15)_0 is surely satisfied by \( G^0(x, t) \) and \( H^0(y, t) \) because \( F(x, 0) = f(x) \).

Next we suppose that \( G^{\mu-1}(x, t) \) and \( H^{\mu-1}(y, t) \) satisfying (A-15)_{\mu-1} are already determined. We rewrite the equation (A-15)_\mu as

(A-16)_\mu \quad F(G^{\mu-1}(x, t) + G_\mu(x, t), t) \equiv H^{\mu-1}(f(x), t) + H_\mu(f(x), t).

Well, we have

\[
F(G^{\mu-1}(x, t) + G_\mu(x, t), t) = F(G^{\mu-1}(x, t), t) + \sum_{\alpha=1}^{n} \frac{\partial F}{\partial x_\alpha} (x, 0) G_\mu(x, t)
\]

\[
= F(G^{\mu-1}(x, t), t) + \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha} (x) G_\mu(x, t),
\]

where \( G_\mu(x, t) = (x_\alpha G_\mu(x, t)) \) for \( 1 \leq \alpha \leq n \).

Hence the equation (A-16)_\mu is equivalent to the following:

(A-17)_\mu \quad F(G^{\mu-1}(x, t), t) + \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha} (x) G_\mu(x, t) \equiv H^{\mu-1}(f(x), t) + H_\mu(f(x), t).

We define a homogeneous polynomial \( G_\mu(x, t) \) in \( (t_1, \ldots, t_i) \) of degree \( \mu \) with coefficients in \( m \)-vector valued holomorphic functions in \( (x_1, \ldots, x_n) \) by the equation of congruence

(A-18)_\mu \quad G_\mu(x, t) \equiv F(G^{\mu-1}(x, t), t) - H^{\mu-1}(f(x), t).

Then by (A-17)_\mu it suffices to show the existence of \( G_\mu(x, t) \) and \( H_\mu(y, t) \) satisfying the following equation for the purpose to prove the existence of \( G^\mu(x, t) \) and \( H^\mu(y, t) \) satisfying the equation of congruence (A-15)_\mu:

(A-19)_\mu \quad \Gamma_\mu(x, t) = - \sum_{\alpha=1}^{n} \frac{\partial f}{\partial x_\alpha} (x) G_\mu(x, t) + H_\mu(f(x), t).
We set
\[ H^\beta_{\mu}(x, t) = (y^\beta H_{\mu})(x, t), \]
\[ F^\beta_{\mu}(x, t) = (y^\beta F_{\mu})(x, t), \]
\[ G^{(\mu), \alpha}_{\nu} = G^{(\mu), \alpha}_{\nu}(C^n \times C^l, p_i \times 0) \longrightarrow (C, G^{(\mu), \alpha}_{\nu}(p_i \times 0)), \]
\[ F^{(\mu), \alpha}_{\nu} = F^{(\mu), \alpha}_{\nu}(C^m \times C^l, p_i \times 0) \longrightarrow (C, F^{(\mu), \alpha}_{\nu}(p_i \times 0)), \]
\[ f^{(i)} = f^{(i)}(C^m, p_i) \longrightarrow (C^m, 0) \text{ and } f^{(i), \beta} = y^\beta f^{(i)} \]
for $1 \leq \beta \leq m$, $1 \leq \alpha \leq n$ and $1 \leq i \leq s$ ($S = \{p_1, \ldots, p_s\}$). We write $F^{(\mu), \beta}_{\nu}$, $G^{(\mu), \alpha}_{\nu}$, and $H^{(\mu), \beta}_{\nu}$ explicitly as follows:
\[ F^{(\mu), \beta}_{\nu}(x, t) = \sum_{v_1 + \cdots + v_{l-1} = \mu} \frac{\partial f^{(i), \beta}}{\partial x_{\alpha}}(x) G^{(\mu), \alpha}_{\nu}(x) f^{(i)}(x) + H^{(\mu), \beta}_{\nu}(y) f^{(i)}(y), \]
for $1 \leq \beta \leq m$, $0 \leq v_0$, $0 \leq v_{l-1} + \cdots + v_1 + \cdots + v_1 = \mu$.

For each multi-index $(v_1, \ldots, v_l)$ with $0 \leq v_1, \ldots, 0 \leq v_0$, and $v_1 + \cdots + v_1 = \mu$ we set
\[ \theta^{(i)}_{\mu v_1, \ldots, v_l}(x) = \sum_{v_{l-1} + \cdots + v_1 = \mu} \gamma^{(i)}_{\mu v_1, \ldots, v_l}(x) \left( \frac{\partial}{\partial y_{\beta}} \right), \]
\[ \xi^{(i)}_{\mu v_1, \ldots, v_l}(x) = \sum_{v_{l-1} + \cdots + v_1 = \mu} \xi^{(i)}_{\mu v_1, \ldots, v_l}(x) \left( \frac{\partial}{\partial x_{\alpha}} \right), \]
\[ \theta^{(i)}_{\mu v_1, \ldots, v_l}(y) = \sum_{v_{l-1} + \cdots + v_1 = \mu} \theta^{(i)}_{\mu v_1, \ldots, v_l}(y) \left( \frac{\partial}{\partial y_{\beta}} \right), \]
\[ \xi^{(i)}_{\mu v_1, \ldots, v_l}(y) = \sum_{v_{l-1} + \cdots + v_1 = \mu} \xi^{(i)}_{\mu v_1, \ldots, v_l}(y) \left( \frac{\partial}{\partial x_{\alpha}} \right). \]

Then $\theta^{(i)}_{\mu v_1, \ldots, v_l}, f^{(i)} \in \Theta_{C^m, 0}$, $\xi^{(i)}_{\mu v_1, \ldots, v_l} \in \Theta_{C^n, S}$, and the equation (A-20)$_{\mu}$ is rewritten as
\[ (A-21)$_{\mu} \quad \theta_{\mu v_1, \ldots, v_l} = \int f^{(i)}(x_{\mu v_1, \ldots, v_l}) + \alpha f^{(i)}(x_{\mu v_1, \ldots, v_l}), \]
for $0 \leq v_1, \ldots, 0 \leq v_0$, $v_1 + \cdots + v_1 = \mu$.

For each $\theta_{\mu v_1, \ldots, v_l}$ there exist surely $\xi_{\mu v_1, \ldots, v_l}$ and $\xi_{\mu v_1, \ldots, v_l}$ satisfying (A-21)$_{\mu}$ because $f$ is simultaneously infinitely stable at $S$. Consequently, going back to the equation (A-19)$_{\mu}$, we conclude that $G^{(\mu), \beta}_{\nu}(x, t)$ ($1 \leq i \leq s$) and $H^{(\mu), \beta}_{\nu}(y, t)$ satisfying (A-19)$_{\mu}$ exist. This shows the equation of congruence (A-15)$_{\mu}$ is surely solved.

II) Proof of convergence:

Now we prove that if we choose the solution $G_{\mu}(x, t)$ and $H_{\mu}(y, t)$ of the equation (A-19)$_{\mu}$ ($\mu \geq 1$) properly in each step of the above construction, the power series
\[ G(x, t) = x + G_{\mu}(x, t) + \cdots + G_{\mu}(x, t) + \cdots; \]
\[ H(y, t) = y + H_{\mu}(y, t) + \cdots + H_{\mu}(y, t) + \cdots \]
converge absolutely and uniformly in a sufficiently small open neighborhoods of $S \times 0$.
and \(0 \times 0\) in \(C^n \times C^l\) and \(C^m \times C^l\), respectively.

Here we introduce a notation. Consider a power series
\[
g(x, t) = \sum g_{h_1, \ldots, h_i}(x) t_1^{k_1} \cdots t_i^{k_i}
\]
whose coefficients \(g_{h_1, \ldots, h_i}(x)\) are \(m\)-vector valued functions, and a power series
\[
a(t) = \sum a_{h_1, \ldots, h_i}(x) t_1^{k_1} \cdots t_i^{k_i}
\]
with non-negative coefficients \(a_{h_1, \ldots, h_i}\). We indicate by writing \(g(x, t) \ll a(t)\) that \(|g_{h_1, \ldots, h_i}(x)| < a_{h_1, \ldots, h_i}\) for any multi-index \((h_1, h_2, \ldots, h_i)\) and any \(x\) with \(1 \leq x \leq m\), where \(g_{h_1, \ldots, h_i}(x)\) denotes the component of \(g_{h_1, \ldots, h_i}(x)\). For the multi-germ \(f: (C^n, S) \rightarrow (C^m, 0)\) of a holomorphic map, we take polydiscs \(\Delta(n; p_i; r_i)\) \((1 \leq i \leq s)\), \(\Delta(m; r)\) in \(C^n\) and \(C^m\), respectively, and a constant \(K\) as in Proposition (B). In the following we denote a representative \(\tilde{f}\) of \(f\) defined in an open neighborhood \(\bigoplus_{i=1}^s \tilde{A}(n; p_i; r_i)\) by the same letter \(f\) for simplicity. We may assume that the unfolding \(F: (C^n \times C^l, S \times 0) \rightarrow (C^m \times C^l, 0 \times 0)\) of \(f\) is defined in an open neighborhood of \(\{ \bigoplus_{i=1}^s \tilde{A}(n; p_i; r_i)\} \times C^l\) and \(\tilde{A}(m; r) \times C^l\), respectively, and that \(A(t)\) holds provided that the constant \(b, c\) are chosen properly. We prove this by induction on \(\mu\).

For \(\mu = 0\), since \(G^0(x, t) = x\) and \(H^0(y, t) = y\), there is nothing to prove.

For \(\mu = 1\), we recall that \(G_1(x, t)\) and \(H_1(y, t)\) are chosen so that the equation
\[
\Gamma(x, t) = \sum_{a=1}^{s} \frac{\partial f}{\partial x_a}(x) G(x, t) + H_1(f(x), t)
\]
is satisfied, where \(\Gamma(x, t)\) is a linear form in \((t_1, \ldots, t_i)\) with coefficients in \(m\)-vector valued holomorphic functions in \((x_1, \ldots, x_n)\) defined by
\[
\Gamma(x, t) = F(x, t) - f(x).
\]
Since \(F(x, t)\) is defined in an open neighborhood of \(\{ \bigoplus_{i=1}^s \tilde{A}(n; p_i; r_i)\} \times C^l\), \(\Gamma_1(x, t)\) is defined in an open neighborhood of \(\{ \bigoplus_{i=1}^s \tilde{A}(n; p_i; r_i)\} \times C^l\) in \(C^n \times C^l\). Therefore \(G_1(x, t)\) (resp. \(H_1(y, t)\)) is defined in an open neighborhood of \(\{ \bigoplus_{i=1}^s \tilde{A}(n; p_i; r_i)\} \times C^l\) in \(C^n \times C^l\) (resp. of \(\tilde{A}(m; r) \times C^l\) in \(C^m \times C^l\)), because \(G_1(x, t)\) and \(H_1(y, t)\) are obtained by solving \((A-20)\), or \((A-21)\), and the polydiscs \(\tilde{A}(n; p_i; r_i)\) \((1 \leq i \leq s)\) and
\( \mathcal{A}(m; r') \) have been chosen so that; for any \( \theta \in f^{*} \theta_{C^{m}}(\bigotimes_{i=1}^{n} \mathcal{A}(n; p_i; r_i)) \) there exist \( \xi \in \Theta_{C^{m}}(\bigotimes_{i=1}^{n} \mathcal{A}(n; p_i; r_i)) \) and \( \zeta \in \Theta_{C^{m}}(\mathcal{A}(m; r')) \) such that \( \theta = tf(\xi) + \omega f(\zeta) \). For the purpose that \((A-23)_{1}\) and \((A-24)_{1}\) hold, it suffices to choose the constant \( b, c \) so that

\[
\begin{align*}
\text{Max}_{t \leq r \leq t'} \| G_t(x) \|_{\mathcal{A}(m; r')} & \leq \frac{b}{16} ; \quad \text{and} \\
\text{Max}_{t \leq r \leq t'} \| H_t(y) \|_{\mathcal{A}(m; r')} & \leq \frac{b}{16},
\end{align*}
\]

where we set \( G_t(x, i) := \sum_{i=1}^{r} G_{t_i}(x) t_i \) and \( H_t(y, i) := \sum_{i=1}^{t} H_{t_i}(y) t_i \).

For \( \mu \geq 2 \), suppose that \( G^{\mu-1}(x, y) \) and \( H^{\mu-1}(y, t) \) are chosen so that they are defined in open neighborhoods of \( \mathcal{A}(n; p_i; r_i) \times C^i \) in \( C^n \times C^i \), and of \( \mathcal{A}(m; r') \times C^i \) in \( C^m \times C^i \), respectively, and suppose that they satisfy \((A-23)_{\mu-1}\) and \((A-24)_{\mu-1}\); and we prove the same hold for \( G^{\mu}(x, t) \) and \( H^{\mu}(y, t) \). We recall that \( G^{\mu}(x, t) = G^{\mu-1}(x, t) + G_{\mu}(x, t) \), \( H^{\mu}(y, t) = H^{\mu-1}(y, t) + H_{\mu}(y, t) \), and that they satisfy the equality

\[
\Gamma_{\mu}(x, t) = -\sum_{a=1}^{n} \frac{\partial f}{\partial x_a} (x) G^{\mu}_{a}(x, t) + H_{\mu}(f(x), t)
\]  

\((A-19)_{\mu}\),

where \( \Gamma_{\mu}(x, t) \) is a homogeneous polynomial in \( (t_1, \ldots, t_l) \) of degree \( \mu \) with coefficients in \( m \)-vector valued holomorphic functions in \( (x_1, \ldots, x_n) \) defined by the equation of congruence

\[
\Gamma_{\mu}(x, t) \equiv F(G^{\mu-1}(x, t), t) - H^{\mu-1}(f(x), t)
\]  

\((A-18)_{\mu}\).

Since \( F(x, t) \), \( G^{\mu-1}(x, t) \), \( H^{\mu-1}(y, t) \) are defined in an open neighborhood of \( \mathcal{A}(n; p_i; r_i) \times C^i \) in \( C^n \times C^i \), and \( H^{\mu-1}(y, t) \) is in an open neighborhood of \( \mathcal{A}(m; r') \times C^i \) in \( C^m \times C^i \), and since \( f(\mathcal{A}(n; p_i; r_i)) \subset (\mathcal{A}(m; r')) \), \( \Gamma_{\mu}(x, t) \) is defined in an open neighborhood of \( \mathcal{A}(n; p_i; r_i) \times C^i \) in \( C^n \times C^i \).

Therefore, recalling how we have determined \( G_{\mu}(x, t) \) and \( H_{\mu}(y, t) \) satisfying \((A-19)_{\mu}\), \( (A-20)_{\mu}\), or \( (A-21)_{\mu}\), we conclude that \( G_{\mu}(x, t) \) is defined in an open neighborhood of \( \mathcal{A}(n; p_i; r_i) \times C^i \) in \( C^n \times C^i \), and \( H_{\mu}(y, t) \) in an open neighborhood of \( \mathcal{A}(m; r') \times C^i \) in \( C^m \times C^i \); so the same hold for \( G^{\mu}(x, t) \) and \( H^{\mu}(y, t) \). For the purpose to prove that \( G^{\mu}(x, t) \) and \( H^{\mu}(y, t) \) satisfy \((A-23)_{\mu}\) and \((A-24)_{\mu}\), we need to estimate \( \Gamma_{\mu}(x, t) \) by a superior power series in \( (t_1, \ldots, t_l) \). We set

\[
A_0(z, t) := \frac{b_0}{c_0} \sum_{i=1}^{\infty} c_i (z_1 + \cdots + z_n + t_1 + \cdots + t_l)^r.
\]

If we take the constants \( b_0, c_0 \) properly, we may assume that the power series expansion of \( F^{\beta}(x + z, t) \) (\( 1 \leq \beta \leq m \)) in \( n + l \) variables \( z_1, \ldots, z_n, t_1, \ldots, t_l \) satisfies

\[
(A-25) \quad F^{\beta}(x + z, t) = f^{\beta}(x) \leq A_0(z, t) \quad (1 \leq \beta \leq m)
\]

for \( x \in \mathcal{A}(n; p_i; r_i) \), where \( F^{\beta} \) and \( f^{\beta} \) denote \( y^{\beta} F \) and \( y^{\beta} f \), respectively. From \( (A-25) \) it follows that

\[
(A-26) \quad F^{\beta}(x + z, t) = f^{\beta}(x) - f^{\beta}(x) - \sum_{a=1}^{\infty} \frac{\partial f^{\beta}}{\partial x_a} (x) z_a
\]
\[ \ll \frac{b_0}{c_0} \sum_{v=2}^{\infty} c_0^v (x_1 + \cdots + z_n + t_1 + \cdots + t_l)^v, \]

(as a power series in \((n + l)\) variables \((z_1, \ldots, z_n, t_1, \ldots, t_l)\) for \(1 \leq \beta \leq m\) and \(x \in \bigoplus_{i=1}^{n} \mathbb{D}(n; p_i; r_i)\)). Substituting \(G^{\mu-1}(x, t) - x\) for \(z\) at \((A-26)\), we have

\[ (A-27) \quad F_\mu(G^{\mu-1}(x, t), t) - f(x) - F_\mu(x, t) - \sum_{a=1}^{n} \frac{\partial f_\mu}{\partial x_a} (x) (G^{\mu-1}(x, t) - x_a) = \frac{b_0}{c_0} \sum_{v=2}^{\infty} c_0^v ((G^{\mu-1}(x, t) - x_1) + \cdots + (G^{\mu-1}(x, t) - x_n) + t_1 + \cdots + t_l)^v \]

for \(x \in \bigoplus_{i=1}^{n} \mathbb{D}(n; p_i; r_i)\) and \(1 \leq \beta \leq m\). If we take the constant \(b\) so that \(\frac{b}{c} > 1\), then we have \(t_j \ll A(t)\) for \(1 \leq j \leq l\). By this and the induction hypothesis \((A-22)_{\mu-1}\), we have

\[ (G^{\mu-1}(x, t) - x_1) + \cdots + (G^{\mu-1}(x, t) - x_n) + t_1 + \cdots + t_l)^v \ll (n + l)^v A(t)^v \]

for \(v \geq 2\). Then by \((A-22)\) we have

\[ (G^{\mu-1}(x, t) - x_1) + \cdots + (G^{\mu-1}(x, t) - x_n) + t_1 + \cdots + t_l)^v \ll (n + l)^v \left(\frac{b}{c}\right)^{v-1} A(t). \]

Hence it follows from \((A-27)\) that

\[ (A-28) \quad F_\mu(G^{\mu-1}(x, t), t) - f(x) - F_\mu(x, t) - \sum_{a=1}^{n} \frac{\partial f_\mu}{\partial x_a} (x) (G^{\mu-1}(x, t) - x_a) = \frac{b_0}{c_0} \sum_{v=2}^{\infty} c_0^v (n + l)^v \left(\frac{b}{c}\right)^v \{ \sum_{v=2}^{\infty} c_0^v (n + l)^v \left(\frac{b}{c}\right)^v \} \}

for \(x \in \bigoplus_{i=1}^{n} \mathbb{D}(n; p_i; r_i)\) and \(1 \leq \beta \leq m\).

Take the constant \(c\) so that \(-c_0(n + l)b/c < 1/2\), then we have

\[ \sum_{v=2}^{\infty} c_0^v (n + l)^v \left(\frac{b}{c}\right)^v = c_0^2 (n + l)^2 \left(\frac{b}{c}\right)^2 \{ \sum_{v=0}^{\infty} c_0^v (n + l)^v \left(\frac{b}{c}\right)^v \} \]

\[ = c_0^2 (n + l)^2 \left(\frac{b}{c}\right)^2 \lim_{v \to \infty} \frac{1 - \left\{ c_0(n + l) \left(\frac{b}{c}\right) \right\}^v}{1 - \left\{ c_0(n + l) \left(\frac{b}{c}\right) \right\}^2} \]

\[ = c_0^2 (n + l)^2 \left(\frac{b}{c}\right)^2 \frac{1}{1 - \left\{ c_0(n + l) \left(\frac{b}{c}\right) \right\}^2}. \]

Hence by \((A-28)\) we have

\[ (A-29) \quad \frac{2(n + l)^2 b b_0 c_0}{c} A(t). \]

Well, since \(\Gamma_\mu(x, t)\) is defined by

\[ \Gamma_\mu(x, t) = F(G^{\mu-1}(x, t), t) - H^{\mu-1}(f(x), t) \]

and \(H^{\mu-1}(f(x), t)\) does not contain the term of degree \(\geq \mu\) as a power series in \(t\), \(\Gamma_\mu(x, t)\) is equal to the term of degree \(\mu\) of \(F(G^{\mu-1}(x, t), t)\) as a power series in \(t\). Hence \(\Gamma_\mu(x, t)\)
is nothing but the term of degree $\mu$ of
\[ F(G^{n-1}(x, t), t) - f(x) - F_t(x, t) - \sum_{a=1}^{n} \frac{\partial f}{\partial x_a}(x) (G^{n-1}(x, t) - x_a) \]
as a power series in $t$, since
\[ f(x) + F_t(x, t) + \sum_{a=1}^{n} \frac{\partial f}{\partial x_a}(x) (G^{n-1}(x, t) - x_a) \]
does not contain the term of degree $\geq \mu$ as a power series in $t$. Therefore, by (A-29) we have
\[(A-30) \quad \Gamma_\mu(x, t) \ll \frac{(n+1)^2bb_0c_0}{c} A(t) \]
for $x \in \bigoplus_{i=1}^{s} \mathcal{A}(n; p_i; r_i)$.

Now, recalling again how we have determined $G_\mu(x, t)$ and $H_\mu(y, t)$ satisfying the equation (A-19)$_\mu$ (cf. (A-20)$_\mu$, or (A-21)$_\mu$), and applying Proposition (B), we derive from (A-30) that
\[ G_\mu(x, t) \ll K \left( \frac{(n+1)^2bb_0c_0}{c} \right) A(t) \quad (x \in \bigoplus_{i=1}^{s} \mathcal{A}(n; p_i; r_i)), \]
\[ H_\mu(y, t) \ll K \left( \frac{(n+1)^2bb_0c_0}{c} \right) A(t) \quad (y \in \mathcal{A}(m; r')), \]
where $K$ is the constant chosen before so that the property (y) in Proposition (B) holds for $f: \bigoplus_{i=1}^{s} \mathcal{A}(n; p_i; r_i) \rightarrow \mathcal{A}(m; r')$. Therefore if we take the constant $c$ sufficiently large so that $K \left( \frac{(n+1)^2bb_0c_0}{c} \right) < 1$, then we have
\[ G_\mu(x, t) \ll A(t) \quad (x \in \bigoplus_{i=1}^{s} \mathcal{A}(n; p_i; r_i)), \]
\[ H_\mu(y, t) \ll A(t) \quad (y \in \mathcal{A}(m; r')), \]
hence by the induction hypothesis (A-23)$_{\mu-1}$ and (A-24)$_{\mu-1}$,
\[ G^\mu(x, t) - x \ll A(t) \quad (x \in \bigoplus_{i=1}^{s} \mathcal{A}(n; p_i; r_i)), \]
\[ H^\mu(y, t) - x \ll A(t) \quad (y \in \mathcal{A}(m; r')). \]
Therefore (A-23)$_{\mu}$ and (A-24)$_{\mu}$ hold for any $\mu \geq 0$. From these it follows that if we set
\[ M_{\frac{1}{c}} = \{ t \in \mathbb{C}^l \mid |t_1| + \cdots + |t_l| < \frac{1}{c} \}, \]
then the power series
\[ G(x, t) = G_0(x, t) + G_1(x, t) + \cdots + G_\mu(x, y) + \cdots; \quad \text{and} \]
\[ H(y, t) = H_0(y, t) + H_1(y, t) + \cdots + H_\mu(y, t) + \cdots \]
converge absolutely and uniformly in $\{ \bigoplus_{i=1}^{s} \mathcal{A}(n; p_i; r) \} \times M_{\frac{1}{c}}$ and $\mathcal{A}(m; r') \times M_{\frac{1}{c}}$, respectively, so they define holomorphic maps defined there.

Furthermore, since $G(x, 0) = x$ and $H(y, 0) = y$, the map $\tilde{G}(x, t) = (G(x, t), t)$ (resp. $\tilde{H}(y, t) = (H(y, t), t)$) give rise a germ of $t$-level preserving analytic automorphism: $(C^n \times C^l, S \times 0) \rightarrow (C^n \times C^l, S \times 0)$ (resp.: $(C^n \times C^l, 0 \times 0) \rightarrow (C^n \times C^l, 0 \times 0)$). Well, by
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(A-15)_{\mu} for \mu = 0, 1, 2, \ldots, we have \(F(G(x, t), t) = H(f(x), t)\). This completes the proof of Theorem (A).

4° Let \(p\) be a point in \(C^n\), \(U\) a domain in \(C^n\) with \(p \in U\), and \(f: U \to C^n\) a holomorphic map. For any \(g \in Hol(U, C^n)\) we set \(h(x) = g(x) - f(x)\) and \(F(x, t) = f(x) + h(x)t\) for \((x, t) \in U \times C\).

**Lemma (B):** Let \(r, r_0\) and \(\delta\) be positive numbers such that \(r < r + \delta < r_0\), 0 < \(\delta < 1\) and \(\Delta(n; p; r) \subset \Delta(n; p; r_0) \subset U\). We set

\[
A_0(z, t) = \frac{b_0}{c_0} \sum_{\mu = 2}^{\infty} c_0^\mu (z_1 + \cdots + z_n)^\mu + \frac{b'_0}{c_0} \sum_{\mu = 1}^{\infty} c_0^\mu (z_1 + \cdots + z_n)^\mu \cdot t
\]

(formal power series in \((z, t))\). If we take the constants \(b_0, c_0, b'_0, c'_0\) so that

1) \(c_0 \geq \frac{1}{\delta}\) and \(b_0 \geq c_0 \|f(x)\|_{\Delta(n; p; r_0)}\); and

2) \(c'_0 \geq \frac{1}{\delta}\) and \(b'_0 \geq c'_0 \|h(x)\|_{\Delta(n; p; r_0)}\),

then we have

\[
F(x + z, t) - \left\{ f(x) + h(x)t + \sum_{\alpha = 0}^{n} \frac{\partial f}{\partial x_\alpha}(x)z_\alpha \right\} \ll A_0(z, t)
\]

for any \(x \in \Delta(n; p; r)\) as a formal power series in \((z, t)\).

**Proof:** Note that \(f(x) + h(x)t + \sum_{\alpha = 0}^{n} \frac{\partial f}{\partial x_\alpha}(x)z_\alpha\) is the term of degree \(\leq 1\) of \(F(x + z, t)\) as a power series in \((z, t)\). Hence we have

\[
F(x + z, t) - \left\{ f(x) + h(x)t + \sum_{\alpha = 0}^{n} \frac{\partial f}{\partial x_\alpha}(x)z_\alpha \right\}
\]

\[
= \sum_{v_1 + \cdots + v_n \geq 2} \sum_{v_1 \geq 0, \ldots, v_n \geq 0} \frac{1}{v_1! \cdots v_n!} \frac{\partial^{v_1 + \cdots + v_n} f}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (x) z_1^{v_1} \cdots z_n^{v_n} t^\mu
\]

\[
= \sum_{v_1 + \cdots + v_n \geq 2} \sum_{v_1 \geq 0, \ldots, v_n \geq 0} \frac{1}{v_1! \cdots v_n!} \frac{\partial^{v_1 + \cdots + v_n} f}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (x) z_1^{v_1} \cdots z_n^{v_n} t^\mu
\]

\[
+ \left( \sum_{v_1 + \cdots + v_n \geq 2} \sum_{v_1 \geq 0, \ldots, v_n \geq 0} \frac{1}{v_1! \cdots v_n!} \frac{\partial^{v_1 + \cdots + v_n} f}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (x) z_1^{v_1} \cdots z_n^{v_n} t^\mu \right) t.
\]

By Cauchy's integral formula, we have

\[
\frac{1}{v_1! \cdots v_n!} \frac{\partial^{v_1 + \cdots + v_n} f}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (x)
\]

\[
= \frac{1}{(2\pi i)^n} \int_{|z_1-x_1|=\delta} \cdots \int_{|z_n-x_n|=\delta} \frac{f(z_1, \ldots, z_n)}{(z_1-x_1)^{v_1+1} \cdots (z_n-x_n)^{v_n+1}} \, dz_1 \cdots dz_n; \text{ and so}
\]

\[
\left| \frac{1}{v_1! \cdots v_n!} \frac{\partial^{v_1 + \cdots + v_n} f}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (x) \right| \leq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \left| \frac{f(x_1 + \delta e^{i\theta_1}, \ldots, x_n + \delta e^{i\theta_n})}{\delta^{v_1+\cdots+v_n}} \right| d\theta_1 \cdots d\theta_n
\]

\[
\leq \|f(z)\|_{\Delta(n; p; r_0)} \leq c_0^{v_1+\cdots+v_n} \|f(z)\|_{\Delta(n; p; r_0)} \leq \frac{b_0}{c_0} c_0^{v_1+\cdots+v_n}
\]

From this it follows that

\[
\sum_{v_1 + \cdots + v_n \geq 2} \sum_{v_1 \geq 0, \ldots, v_n \geq 0} \frac{1}{v_1! \cdots v_n!} \frac{\partial^{v_1 + \cdots + v_n} f}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (x) z_1^{v_1} \cdots z_n^{v_n} \ll \frac{b_0}{c_0} \sum_{\mu = 2}^{\infty} c_0^\mu (z_1 + \cdots + z_n)^\mu.
\]
Similarly, we have
\[ \sum_{v_1 + \cdots + v_n = 1} \frac{1}{v_1! \cdots v_n!} \frac{\partial^{v_1 + \cdots + v_n} h}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} (x) z_1^{v_1} \cdots z_n^{v_n} \ll \frac{b_0}{c_0} \sum_{\mu=1}^{\infty} c^\mu (z_1 + \cdots + z_n)^\mu. \]

Consequently, we have
\[ F(x + z, t) - \left\{ f(x) + h(x)t + \sum_{a=1}^{n} \frac{\partial f}{\partial x_a} (x) z_a \right\} \ll A_0(z, t_0) \]
for \( x \in \tilde{A}(n; p; r) \). Q.E.D.

Let \( U_1, \ldots, U_s \) be domains in \( C^n \) which are mutually disjoint. We set \( U = \bigoplus_{i=1}^{s} U_i \), i.e., the disjoint union of \( U_1, \ldots, U_s \). Let \( S = \{p_1, \ldots, p_s\} \) be a finite subset of \( C^n \) with \( p_i \in U_i \) for \( 1 \leq i \leq s \). With these notations we have the following theorem.

**Theorem (B):** Let \( f: U \to C^n \) be a holomorphic map with \( f(S) = 0 \). If \( f \) is simultaneously stable at \( S \), then for any relatively compact open neighborhood \( U' \) of \( S \) in \( U \) of the form \( U' = \bigoplus_{i=1}^{s} \Delta(n; p_i; r_0) \), i.e., a disjoint union of polydiscs \( \Delta(n; p_i; r_0) \subseteq U \) with the center \( p_i \) (\( 1 \leq i \leq s \)), there exists an open neighborhood
\[ N_0(U') = \{ g \in H^1(U, C^m) | \sup_{x \in \partial U'} |f(x) - g(x)| < \varepsilon \} \]
of \( f \) in \( H^1(U, C^m) \) which has the following property, where \( \varepsilon \) is a suitably chosen positive number:

for any \( g \in N_0(U') \) there exists a quadruple \( (U'', W, \phi, \psi) \) where \( U'' = \bigoplus_{i=1}^{s} U_i \) is a disjoint union of domains \( U_i \subseteq \Delta(n; p_i; r_0) \) with \( p_i \in U_i \) (\( 1 \leq i \leq s \)), \( W \) an open neighborhood of \( 0 \) in \( C^m \), \( \phi \) an analytic embedding (i.e. invertible holomorphic map): \( U'' \to U' \) of the form \( \phi = \bigoplus_{i=1}^{s} \phi_i \), i.e., a disjoint sum of analytic embeddings \( \phi_i: U_i \to \Delta(n; p_i; r_0) \) (\( 1 \leq i \leq s \)), and \( \psi \) an analytic embedding: \( W \to C^m \), such that the following diagram commutes:

\[ \begin{array}{ccc} U'' & \xrightarrow{\phi} & U \\ \downarrow f|_{U''} & & \downarrow \phi \\ W & \xrightarrow{\psi} & C^m \end{array} \]

**Proof:** In the following we shall use the same notations and terminology as in the proof of Theorem (A). Let \( g \in H^1(U, C^m) \). We set \( h(x) = g(x) - f(x) \) and \( F(x, t) = f(x) + h(x)t \) for \( (x, t) \in U \times C \). We regard \( F \) as an unfolding of \( f \) parametrized by \( C \).

Then, as we have seen in the proof of Theorem (A), there exist a disjoint union \( \bigoplus_{i=1}^{s} \Delta(n; p_i; r) \) of polydiscs \( \Delta(n; p_i; r) \subseteq \Delta(n; p_i; r_0) \) (\( 1 \leq i \leq s \)), polydiscs \( \Delta(m; r') \) and \( \Delta(1; r') \) in \( C^m \) and \( C \), respectively, and \( t \) level preserving analytic embeddings
\[ G(x, t) := (G(x, t): (\bigoplus_{i=1}^{s} \Delta(n; p_i; r)) \times \Delta(1; r') \to C^n \times C, \]
\[ H(y, t) := (H(y, t, l): \Delta(m; r') \times \Delta(1; r') \to C^m \times C \]
such that:

1) \( F((\bigoplus_{i=1}^{s} \Delta(n; p_i; r)) \times \Delta(1; r')) \subseteq \Delta(m; r') \);
(A–31) 2) \( F(\mathbf{x}, t) = H(f(\mathbf{x}), t) \quad \text{for} \quad (\mathbf{x}, t) \in \left\{ \bigoplus_{i=1}^{i=n} A(n; p_i; r) \right\} \times A(1; r'''); \)

3) \( G(\mathbf{x}, 0) = \mathbf{x}, \quad H(y, 0) = y. \)

In the subsequence we will prove that there exists a positive number \( \varepsilon \) such that if \( \| g - f \|_{\mathcal{U}_1} := \sup_{x \in \mathcal{U}_1} |g(x) - f(x)| < \varepsilon \), then we can take the positive number \( r'' \) above equal to 2. If this is proved, \( G_1(\mathbf{x}) := G(\mathbf{x}, 1), \) (resp. \( H_1(y) := H(y, 1) \)) give rise to an analytic embedding defined in an open neighborhood of \( S \) in \( C^n \) (resp. in an open neighborhood of the origin of \( C^m \)); and since \( F(\mathbf{x}, 1) = g(\mathbf{x}) \), we have \( g(G_1(\mathbf{x})) = H_1(f(\mathbf{x})) \) in an open neighborhood of the origin of \( C^n \) by the property (2) above at (A–31). Thus the proof of the theorem will be completed.

To prove the existence of such a positive number \( \varepsilon \), we must modify slightly the proof of Theorem (A). We set

\[
A(t) = \frac{b}{\delta} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \left( \frac{t}{2} \right)^\mu.
\]

Note that this power series in \( t \) converges absolutely and uniformly in \( D_2 = \{ t \in C | |t| < 2 \} \).

By induction on \( \mu \), we will prove that if \( \| h \|_{\mathcal{U}_1} = \| g - f \|_{\mathcal{U}_1} \) is sufficiently small and the constant \( b \) above is properly chosen, then there exist \( G_\mu(\mathbf{x}, t) \) and \( H_\mu(y, t) \) for \( \mu = 0, 1, 2, \ldots \) such that:

(A–32) \( F(G_\mu(\mathbf{x}, t), t) = H_\mu(f(\mathbf{x}), t) \);

(A–33) \( G_\mu(\mathbf{x}, t) = x \ll A(t) \quad \text{for} \quad x \in \bigoplus_{i=1}^{i=n} \mathcal{A}(n; p_i; r) \);

(A–34) \( H_\mu(y, t) = y \ll A(t) \quad \text{for} \quad y \in \mathcal{A}(m; r') \).

As we have seen in the proof of Theorem (A), we can take \( G_\mu(\mathbf{x}, t) \) (resp. \( H_\mu(y, t) \)) satisfying (A–32) \( \mu \) so that it is a polynomial in \( t \) of degree \( \mu \) with coefficients in \( n \)-vector valued (resp. \( m \)-vector valued) holomorphic functions in \( x \) (resp. in \( y \)) defined in an open neighborhood of \( \bigoplus_{i=1}^{i=n} \mathcal{A}(n; p_i; r) \) (resp. \( \mathcal{A}(m; r') \)). Our task is to prove the estimate

(A–33) \( \mu \) and (A–34) \( \mu \).

For \( \mu = 0 \) there is nothing to prove.

For \( \mu = 1 \), if we write \( G_1(\mathbf{x}, t) = G_1(\mathbf{x}) t \) and \( H_1(y, t) = H_1(y) t \), then \( G_1(\mathbf{x}) \) and \( H_1(y) \) are obtained as the solutions of the equation:

\[
- h(x) = \sum_{\mu=1}^{\infty} \frac{\partial f}{\partial x_\mu} (x) G_1(x) + H_1(f(x)),
\]

because we have \( \Gamma_1(\mathbf{x}, t) = h(x) t \) in this case. Hence by Proposition (B) we have estimates:

\[
\| G_1(x) \|_{\mathcal{O}(n, p_i; r)} \leq K \| h(x) \|_{\mathcal{O}(n, p_i; r)} \quad \text{and}
\]

\[
\| H_1(y) \|_{\mathcal{O}(m, r')} \leq K \| h(x) \|_{\mathcal{O}(n, p_i; r)},
\]

where \( K \) is a constant which depends only on \( f \). Therefore, if

(A–35) \( \| h(x) \|_{\mathcal{O}(n, p_i; r)} \leq \frac{b}{16 K} \),

then (A–33), and (A–34) \( 1 \) surely hold.

For \( \mu \geq 2 \), suppose that \( G^{\mu - 1}(\mathbf{x}, t) \) and \( H^{\mu - 1}(y, t) \) satisfy (A–33) \( \mu - 1 \) and (A–34) \( \mu - 1 \),
and we prove that $G^u(x, t)$ and $H^u(y, t)$ satisfy (A–33)$_2$ and (A–34)$_2$. For this purpose, we need to estimate $\Gamma^u(x, t)$.

Let $\delta$ be a positive number such that $r < r + \delta < r_0$, $0 < \delta < 1$. Then, by Lemma (B), if the following conditions are satisfied:

\begin{align}
(A-36) & \quad c_0 \geq \frac{1}{\delta} \quad \text{and} \quad b_0 \geq c_0 \| f(x) \|_{\ell^1_{\delta}} d(r, p; r_0), \\
(A-37) & \quad c_0' \geq \frac{1}{\delta} \quad \text{and} \quad b_0' \geq c_0' \| h(x) \|_{\ell^1_{\delta}} d(r, p; r_0),
\end{align}

we have

\begin{equation}
\frac{F(x+z, t) - \left\{ f(x) + h(x) t + \sum_{\alpha=1}^{\infty} \frac{\partial f}{\partial x_{\alpha}}(x)z_{\alpha} \right\}}{A_0(z, t)} \ll A_0(z, t)
\end{equation}

for any $x \in \mathcal{A}(n; p; r)$ as a formal power series in $(z, t)$, where $A_0(z, t)$ denotes the same as in Lemma (B). Substituting $G^{u-1}(x, t) - x$ for $z$ at (A–38), we have

\begin{equation}
\frac{F(G^{u-1}(x, t), t) - \left\{ f(x) + h(x) t + \sum_{\alpha=1}^{\infty} \frac{\partial f}{\partial x_{\alpha}}(x)(G^{u-1}(x, t) - x_{\alpha}) \right\}}{A_0(I, t)} \ll \frac{b_0}{c_0} \sum_{\mu=1}^{\infty} c_0^{\mu} ((G^{1-u-1}(x, t) - x_{\alpha}) + \cdots + (G^{u-u-1}(x, t) - x_{\alpha}))^\mu
\end{equation}

\begin{equation}
+ \frac{b_0'}{c_0'} \sum_{\mu=1}^{\infty} c_0'^{\mu} ((G^{1-u-1}(x, t) - x_{\alpha}) + \cdots + (G^{u-u-1}(x, t) - x_{\alpha}))^\mu.
\end{equation}

By induction hypothesis, $G^{u-u-1}(x, t) - x \ll A(t)$ $(1 \leq \alpha \leq n)$; hence we have

**Right hand side of (A–39)**

\begin{equation}
\frac{b_0}{c_0} \sum_{\mu=1}^{\infty} c_0^{\mu} n^\mu A(t)^{\mu} + \left( \frac{b_0'}{c_0'} \sum_{\mu=1}^{\infty} c_0'^{\mu} n^\mu A(t)^{\mu} \right) t
\end{equation}

\begin{equation}
\ll \frac{b_0}{c_0} \sum_{\mu=1}^{\infty} c_0^{\mu} n^\mu (2b)^{\mu-1} A(t) + b_0 n A(t) + \left( \frac{b_0'}{c_0'} \sum_{\mu=1}^{\infty} c_0'^{\mu} n^\mu (2b)^{\mu-1} A(t) \right) t,
\end{equation}

where the second $\ll$ follows from $A(t)^{\mu} \ll (2b)^{\mu-1} A(t)$ $(\mu \geq 2)$ (cf. (A–22)).

We claim that

\begin{equation}
A(t) t \ll 8 A(t),
\end{equation}

which can be seen as follows:

since

\begin{equation}
A(t) t = \frac{b}{8} \sum_{\mu=1}^{\infty} \frac{1}{\mu^2} \left( \frac{1}{2} t \right)^{\mu-1} = \frac{b}{8} \sum_{\mu=2}^{\infty} \left( \frac{1}{\mu-1} \right)^{\mu-1} t^\mu,
\end{equation}

and

\begin{equation}
8 A(t) = b \sum_{\mu=1}^{\infty} \left( \frac{1}{2} t \right)^{\mu-1},
\end{equation}

(A–41) is equivalent to

\begin{equation}
\frac{b}{8} \left( \frac{1}{2} \right)^{\mu-1} \leq \frac{b}{8} \left( \frac{1}{2} \right)^{\mu-1} \quad (\mu \geq 2),
\end{equation}

which surely holds, because $\left( \frac{\mu}{\mu-1} \right)^2 \leq 4$ for $\mu \geq 2$. By (A–41) we have

**Right hand side of (A–40)**

\begin{equation}
\ll \left\{ \frac{b_0}{2bc_0} \sum_{\mu=2}^{\infty} (2nbc_0)^{\mu} + 8b_0 n + \frac{4b_0'}{bc_0} \sum_{\mu=2}^{\infty} (2nbc_0)^{\mu} \right\} A(t).
\end{equation}

If we take the constant $b$ sufficiently small so that
(A-43) \( 2nb_0c_0 < \frac{1}{2} \) and \( 2nb' c_0 < \frac{1}{2} \),
then we have

(A-44) \( \sum_{\mu=2}^{\infty} (2nb_0 c_0)^\mu < 8(nbc_0)^2 \),

(A-45) \( \sum_{\mu=2}^{\infty} (2nb' c_0)^\mu < 8(nbc_0)^2 \).

Therefore by (A-39), (A-40), (A-42), (A-44) and (A-45), we have

\[
F(G^{\mu-1}(x, t), t) - \{ f(x) + h(x)t + \sum_{a=1}^{\infty} \frac{\partial f}{\partial x_a}(x)(G^{\alpha, \mu-1}(x, t) - x_a) \}
\]

(A-46) \( \ll \left\{ \frac{b_0}{2bc_0} 8(nbc_0)^2 + 8b'_0n + \frac{4b'_0}{bc_0} 8(nbc_0' c_0)^2 \right\} A(t) \)

\( = (b(4b_0 n^2 c_0 + 32b' n^2 c_0' + 8b'_0 n) A(t)) \).

Note that \( \Gamma_\mu(x, t) (\mu \geq 2) \) is nothing but the term of degree \( \mu \) of the left hand side of
(A-46), because \( \Gamma_\mu(x, t) \) has the form \( \Gamma_\mu(x)^\mu \), where \( \Gamma_\mu(x) \) is a \( m \)-vector valued holomorphic function defined in an open neighborhood of \( \bigoplus_{i=1}^{\infty} \bar{A}(n; p_i; r) \), and because

\( \Gamma_\mu(x, t) \) is defined by

\( \Gamma_\mu(x, t) = F(G^{\mu-1}(x, t), t) - H^{\mu-1}(f(x), t) \).

Hence we have

(A-47) \( \Gamma_\mu(x, t) \ll \left\{ b(4b_0 n^2 c_0 + 32b'_0 n^2 c'_0 + 8b'_0 n) A(t) \right\} \) \( (x \in \bigoplus_{i=1}^{\infty} \bar{A}(n; p_i; r)) \)

for \( \mu \geq 2 \).

Therefore, recalling how we have determined \( G_\mu(x, t) \) and \( H_\mu(y, t) \) (cf. (A-20)\( \mu \), or
(A-21)\( \mu \)), we have

\( G_\mu(x, t) \ll K \left\{ b(4b_0 n^2 c_0 + 32b'_0 n^2 c'_0 + 8b'_0 n) A(t) \right\} \) \( (x \in \bigoplus_{i=1}^{\infty} \bar{A}(n; p_i; r)) \), and

\( H_\mu(y, t) \ll K \left\{ b(4b_0 n^2 c_0 + 32b'_0 n^2 c'_0 + 8b'_0 n) A(t) \right\} \) \( (y \in \bar{A}(m; r')) \)

by Proposition (B), where \( K \) is a constant which does not depend on \( \mu \). From these it follows that, if we take the constant \( b, b'_0 \) sufficiently small so that

(A-48) \( Kb(4b_0 n^2 c_0 + 32b'_0 n^2 c'_0) < \frac{1}{2} \), and

(A-49) \( 8Kb'_0 n < \frac{1}{2} \),

then we have \( G_\mu(x, t) \ll A(t) \) for \( x \in \bigoplus_{i=1}^{\infty} \bar{A}(n; p_i; r) \) and \( H_\mu(y, t) \ll A(t) \) for \( y \in \bar{A}(m; r') \); and so

\( G^\mu(x, t) - x \ll A(t) \) \( \quad \text{for} \quad x \in \bigoplus_{i=1}^{\infty} \bar{A}(n; p_i; r) \), and

\( H^\mu(y, t) - y \ll A(t) \) \( \quad \text{for} \quad y \in \bar{A}(m; r') \).

Hence (A-33)\( \mu \) and (A-34)\( \mu \) hold for any \( \mu \geq 0 \).

Well, note that \( A(t) \) converges in \( M_2 := \{ t \in C \mid | t | < 2 \} \). Therefore, by (A-33)\( \mu \) and
(A-34)\( \mu \) for \( \mu = 0, 1, 2, \cdots \), we infer that the power series

\( G(x, t) = x + G_1(x, t) + \cdots + G_\mu(x, t) + \cdots \)
and \[ H(y, t) = y + H_1(y, t) + \cdots + H_\mu(y, t) + \cdots \]

converge absolutely uniformly in \( \bigcup_{i=1}^g \Delta(n; p_i; r) \times M_2 \) and \( \Delta(m; r') \times M_2 \), respectively; so they define holomorphic maps defined there.

Furthermore, by (A–32)\( _\mu \) for \( \mu = 0, 1, 2, \cdots \), we have
\[ F(G(x, t), t) = H(f(x), t). \]

Since \( F(x, 1) = g(x) \), substituting \( 1 \) for \( t \) of this identity, one obtains
\[ g(G_1(x)) = H_1(f(x)) \quad (x \in \bigcup_{i=1}^g \Delta(n; p_i; r)), \]
where \( G_1(x) := G(x, 1) \) and \( H_1(y) := H(y, 1) \). On the other hand, setting \( t = 1 \) at (A–33)\( _\infty \) and (A–34)\( _\infty \), one has
\[ G_1(x) - x \quad \text{and} \quad H_1(y) - y \ll A(1) = \frac{8}{b} \sum_{\mu = 1}^\infty \frac{1}{\mu^2} \left( \frac{1}{2} \right)^\mu. \]

Hence if we take the constant \( b \) sufficiently small, we may conclude that \( G_1(x) \) (resp. \( H_1(y) \)) give rise an analytic embedding \( \phi \) from a sufficiently small open neighborhood \( U' \) of \( S \) in \( \bigcup_{i=1}^g \Delta(n; p_i; r) \) into \( U_1 \) (resp. an analytic embedding \( \psi \) from a sufficiently small open neighborhood \( W \) of \( 0 \) in \( C^m \) into \( C^m \)).

Here we summarize how we have chosen the constants \( b_0, c_0, b'_0, c'_0 \) and \( b \) to make sure of that there are no contradictions among these choices.

1) first, we choose the constants \( b_0 \) and \( c_0 \) so that;
\[ c_0 \geq \frac{1}{\delta} \quad \text{and} \quad b_0 \geq c_0 \| f(x) \| \quad \text{(cf. (A–36))}; \]

2) second, we choose the constants \( b'_0 \) and \( c'_0 \) so that;
\[ 8Kb'_0n < \frac{1}{2} \quad \text{and} \quad c'_0 \geq \frac{1}{\delta} \quad \text{(cf. (A–49), (A–37))}; \]

3) finally, we choose the constant \( b \) so that;
\[ (a) \quad 2nbc_0 < \frac{1}{2} \quad \text{and} \quad 2nb'_0 < \frac{1}{2} \quad \text{(cf. (A–43))}; \]
\[ (b) \quad Kb(4b_0n^2c_0 + 32b'_0n^2c'_0) < \frac{1}{2} \quad \text{(cf. (A–48))}; \]
\[ (c) \quad \text{for any germs } \phi': (C^n, S) \rightarrow (C^n, \phi'(S)) \quad \text{and} \quad \psi': (C^m, 0) \rightarrow (C^m, \psi'(0)) \]
\[ \text{of homomorphic maps, if they satisfy} \]
\[ \phi'(x) - x \quad \text{and} \quad \psi'(y) - y \ll A(1) = \frac{b}{8} \sum_{\mu = 1}^\infty \frac{1}{\mu^2} \left( \frac{1}{2} \right)^\mu, \]
\[ \text{then they give rise to germs of analytic embeddings (i.e., invertible homomorphic maps).} \]

By the facts proved till now, we conclude that if we set
\[ \varepsilon = \min \left( \frac{b'_0}{c'_0}, \frac{b}{16K} \right) \]
and \[ N_{\delta(U')} = \{ g \in Hol(U, C^m) \mid \| g - f \| \leq \varepsilon \} \]
(cf. (A–35), (A–37)), then the assertions of Theorem (B) surely hold. This completes of the proof of Theorem (B).
References


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