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A Certain Degenerate Ordinary Singularity of Dimension Three * †

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Abstract We give an example of a three dimensional non-isolated singularity which might be considered as a *degenerate* ordinary triple point of dimension three. We show that its normalization is a rational isolated singularity with multiplicity 4, and is *rigid* under deformations.

1. THREE DIMENSIONAL ORDINARY SINGULARITIES

Definition 1. The *ordinary singularities of dimension three* are defined to be one of the following germs of three dimensional hypersurface singularities at the origin of $\mathbf{C}^4$:

$$\begin{cases} (i)\ w=0 \ \text{(simple point)} & (ii)\ zw=0\ \text{(ordinary double point)} \\ (iii)\ yzw=0\ \text{(ordinary triple point)} & (iv)\ xyzw=0\ \text{(ordinary quadruple point)} \\ (v)\ xy^2-z^2=0\ \text{(cuspidal point)} & (vi)\ w(xy^2-z^2)=0\ \text{(stationary point)}, \end{cases}$$

where (x,y,z,w) is the coordinate on $\mathbf{C}^4$ (cf. [1]).

It is known that every non-singular complex projective threefold X is birationally equivalent to a hypersurface Y with ordinary singularities in the complex projective 4-space $\mathbf{P}^4(\mathbf{C})$, and that a birational morphism from X to Y is obtained as the composite of an embedding $X \to \mathbf{P}^N(\mathbf{C})$ of X to a complex projective space of sufficiently high dimension and a generic linear projection $\mathbf{P}^N(\mathbf{C}) \to \mathbf{P}^4(\mathbf{C})$.

2. AN EXAMPLE OF A THREE DIMENSIONAL COMPLEX PROJECTIVE HYPERSURFACE WITH ORDINARY SINGULARITIES

Let H_i $(1 \le i \le 4)$ be non-singular hypersurfaces of degrees r_i $(1 \le i \le 4)$, respectively, in the complex projective 4-space $\mathbf{P}^4(\mathbf{C})$ such that they are in general position at every point where they intersect. We put $D_T^{(ij)} := H_i \cap H_j$ $(1 \le i < j \le 4)$ and $D_T := \bigcup_{1\le i<j\le 4} D_T^{(ij)}$. Let f_i $(1 \le i \le 4)$ be the homogeneous polynomial of degree r_i which defines the hypersurface H_i. We may assume $r_1 \ge r_2 \ge r_3 \ge r_4$ because of symmetry. We choose and fix a positive

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integer n with $n \geq 2r_1 + 2r_2 + 2r_3$. Let T be a hypersurface in $\mathrm{P}^4(\mathbf{C})$ defined by the equation

$$F := Af_1f_2f_3f_4 + B(f_1f_2f_3)^2 + C(f_1f_2f_4)^2 + D(f_1f_3f_4)^2 + E(f_2f_3f_4)^2 = 0, \tag{2.1}$$

where A, B, C, D and E are homogeneous polynomials of five variables of respective degrees $n - r_1 - r_2 - r_3 - r_4$, $n - 2r_1 - 2r_2 - 2r_3$, $n - 2r_1 - 2r_2 - 2r_4$, $n - 2r_1 - 2r_3 - 2r_4$ and $n - 2r_2 - 2r_3 - 2r_4$. By Bertini's theorem, T is non-singular outside D_T if we choose sufficiently generic A, B, C, D and E.

Proposition 1. *If the homogeneous polynomials A, B, C, D and E are chosen sufficiently generic, then T has every ordinary singularity except a stationary point, together with the following one:*

$$(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0 \tag{2.2}$$

where (x, y, z, w) is the coordinate on $\mathbf{C}^4$.

Proof. (i) Let $p \in D_T$ be a point satisfying $f_i(p) = 0, 1 \leq i \leq 4$. We may assume that $A(p)B(p)C(p)D(p)E(p) \neq 0$. We make the transformations of local coordinates

$$(f_1, f_2, f_3, f_4) \to \left(\sqrt[4]{\tfrac{A^2E}{BCD}}\tfrac{X}{\sqrt{1+f}}, \sqrt[4]{\tfrac{A^2D}{BCE}}\tfrac{Y}{\sqrt{1+f}}, \sqrt[4]{\tfrac{A^2C}{BDE}}\tfrac{Z}{\sqrt{1+f}}, \sqrt[4]{\tfrac{A^2B}{CDE}}\tfrac{W}{\sqrt{1+f}}\right),$$

where

$$f := (XY)^2 + (XZ)^2 + (XW)^2 + (YZ)^2 + (YW)^2 + (ZW)^2 \\ + XYZW(X^2 + Y^2 + Z^2 + W^2 + XYZW),$$

and

$$(X + YZW, Y + XZW, Z + XYW, W + XYZ) \to (X', Y', Z', W')$$

successively in a neighborhood of p. Then the equation in (2.1) is transformed to $A'X'Y'Z'W' = 0$, where $A' := A^3/\{\sqrt{BCDE}(1 + f)^3\}$. Namely, the point p is an ordinary quadruple point.

(ii) Let $p \in D_T$ be a point where three of $f_i, i = 1, 2, 3, 4$, vanish, but all of $f_i, i = 1, 2, 3, 4$, do not. Suppose that $f_1(p) = f_2(p) = f_3(p) = 0$ and $f_4(p) \neq 0$. We write F in (2.1) as

$$F = A'f_1f_2f_3 + C'(f_1f_2)^2 + D'(f_1f_3)^2 + E'(f_2f_3)^2 \tag{2.3}$$

where $A' := Af_4 + Bf_1f_2f_3$, $C' := Cf_4^2$, $D' := Df_4^2$ and $E' := Ef_4^2$. We may assume that both of A' and $C'D'E'$ do not vanish at p.

(ii-α) In the case of $A'(p)C'(p)D'(p)E'(p) \neq 0$: We make the transformations of local coordinates

$$(f_1, f_2, f_3) \to \left(\tfrac{A'}{\sqrt{C'D'}}\tfrac{X}{1+g}, \tfrac{A'}{\sqrt{C'E'}}\tfrac{Y}{1+g}, \tfrac{A'}{\sqrt{D'E'}}\tfrac{Z}{1+g}\right),$$

where $g := X^2 + Y^2 + Z^2 + XYZ$, and

$$(X+YZ, Y+XZ, Z+XY) \to (X', Y', Z')$$

successively in a neighborhood of p. Then the equation $F = 0$ is transformed to $A''X'Y'Z' = 0$, where $A'' := A'^4/\{C'D'E'(1+g)^4\}$. Hence, p is an ordinary triple point.

(ii-β) In the case of $A'(p) \neq 0$, $C'(p)D'(p)E'(p) = 0$: Taking sufficiently generic C, D and E, we may assume that all of C', D' and E' do not vanish at p. Suppose that $C'(p) = 0$ and $D'(p)E'(p) \neq 0$. We put $X := f_1$, $Y := f_2$, $Z := f_3$ and $W := C'$. We may consider that (X, Y, Z, W) is a system of local coordinates at p by taking a sufficiently generic C. Using the local coordinate (X, Y, Z, W), we can write F in (2.3) as

$$F = A'XYZ + W(XY)^2 + D'(XZ)^2 + E'(YZ)^2 \tag{2.4}$$

where $A'(p)D'(p)E'(p) \neq 0$. We make the transformations of local coordinates

$$(X, Y, Z, W) \to (\tfrac{A'}{\sqrt{D'}}\tfrac{X'}{1+h}, \tfrac{A'}{\sqrt{E'}}\tfrac{Y'}{1+h}, \tfrac{A'}{\sqrt{D'E'}}\tfrac{Z'}{1+h}, W)$$

where $h := Z'^2 + (X'^2 + Y'^2 + X'Y'Z')W$, and

$$(X' + Y'Z', Y' + X'Z', Z' + X'Y'W, W) \to (X'', Y'', Z'', W)$$

successively in a neighborhood of p. Then the equation $F = 0$ is transformed to $A''X''Y''Z'' = 0$, where $A'' := A'^4/\{D'E'(1+h)^4\}$. Hence, p is an ordinary triple point.

(ii-γ) In the case of $A'(p) = 0$, $C'(p)D'(p)E'(p) \neq 0$: We put $X := f_1$, $Y := f_2$, $Z := f_3$ and $W := A'$. We may consider that (X, Y, Z, W) is a system of local coordinates at p by taking sufficiently generic A and B. Using the local coordinate (X, Y, Z, W), we can write F in (2.3) as

$$F = XYZW + C'(XY)^2 + D'(XZ)^2 + E'(YZ)^2. \tag{2.5}$$

We make the transformation of local coordinates

$$(X, Y, Z, W) \to (\tfrac{X'}{\sqrt{C'D'}}, \tfrac{Y'}{\sqrt{C'E'}}, \tfrac{Z'}{\sqrt{D'E'}}, W).$$

Then the equation $F = 0$ is transformed to

$$\frac{1}{C'D'E'}\{X'Y'Z'W + (X'Y')^2 + (X'Z')^2 + (Y'Z')^2\} = 0$$

which defines the singularity in (2.2).

(iii) Let $p \in D_T$ be a point where two of $f_i, i = 1,2,3,4$, vanish, but more than two of $f_i, i = 1,2,3,4$, do not. Suppose that $f_1(p) = f_2(p) = 0$ and $f_3(p)f_4(p) \neq 0$. We write F in (2.1) as

$$F = B'f_1^2 + A'f_1f_2 + E'f_2^2, \tag{2.6}$$

where $B' := (Bf_3^2 + Cf_4^2)f_2^2 + Df_3^2f_4^2$, $A' := Af_3f_4$ and $E' := Ef_3^2f_4^2$.

(iii-α) In the case of $B'(p) \neq 0$, or $E'(p) \neq 0$: Suppose $B'(p) \neq 0$. Then F in (2.6) is written as

$$F = B'(f_1 + \tfrac{A' - \sqrt{A'^2 - 4B'E'}}{2B'} f_2)(f_1 + \tfrac{A' + \sqrt{A'^2 - 4B'E'}}{2B'} f_2)$$

in a neighborhood of p.

(iii-α)$_d$ If $(A'^2 - 4B'E')(p) \neq 0$, then the transfomation

$$f_1 + \tfrac{A' - \sqrt{A'^2 - 4B'E'}}{2B'} f_2 \longrightarrow X,$$

$$f_1 + \tfrac{A' + \sqrt{A'^2 - 4B'E'}}{2B'} f_2 \longrightarrow Y$$

can be regarded as that of local coordinates. By this trasformation the equation $F = 0$ is transformed to $B'XY = 0$, where B' is a non-vanishing factor. Hence p is an ordinary double point.

(iii-α)$_c$ If $(A'^2 - 4B'E')(p) = 0$, we make the transformation of local coordinates

$$\tfrac{A'^2 - 4B'E'}{(2B')^2} f_2 \longrightarrow X,$$

$$f_2 \longrightarrow Y,$$

$$f_1 + \tfrac{A'}{2B'} f_2 \longrightarrow Z$$

in a neighborhood of p. Then the equation $F = 0$ is transformed to

$$B'(Z + \sqrt{X}Y)(Z - \sqrt{X}Y) = B'(Z^2 - XY^2) = 0.$$

Hence p is a cuspidal point.

(iii-β) In the case of $B'(p) = E'(p) = 0$: We put $X := f_1, Y := f_2, Z := B'$ and $W = E'$. We may consider that (X, Y, Z, W) is a system of a local coordinates at p by taking sufficiently generic B, C, D and E. Using the local coordinate (X, Y, Z, W), we can write F in (2.6) as

$$F = A'XY + ZX^2 + WY^2. \tag{2.7}$$

We may assume that $A'(p) \neq 0$. We make the transformations of local coordinates

$$(X, Y, Z, W) \to (X, Y, \tfrac{Z'}{A'}, \tfrac{W'}{A'}),$$

$$(X, Y, Z', W') \to (\tfrac{X'}{1+Z''W''}, \tfrac{Y'}{1+Z''W''}, \tfrac{Z''}{1+Z''W''}, \tfrac{W''}{1+Z''W''}), \text{ and}$$

$$(X' + W''Y', Y' + Z''X', Z'', W'') \longrightarrow (X'', Y'', Z'', W'')$$

successively in a neighborhood of p. Then the equation $F = 0$ is transformed to $A''X''Y'' = 0$, where $A'' := A'/(1 + Z''W'')^3$. Hence p is an ordinary double point.

Q.E.D.

3. THE SINGULARITY $(XY)^2+(YZ)^2+(ZX)^2+XYZW=0$

Proposition 2. *In the expression $(xy)^2+(yz)^2+(zx)^2+xyzw=0$ above, we consider w as parameter. Then, if $w\neq 0$, the singularity defined by this equation is an ordinary triple point.*

Proof. The equation $(xy)^2+(yz)^2+(zx)^2+xyzw=0$ is a special one of the equation $F=0$ in the case (ii-α) above if $w\neq 0$. Hence it defines an ordinary triple point around $(0,0,0,w)$ with $w\neq 0$.

Q.E.D.

Due to Proposition 2, the singularity $(xy)^2+(yz)^2+(zx)^2+xyzw=0$ might be considerd as a *degenerate* ordinary triple point.

Proposition 3. *Let $v:\mathrm{P}^2(\mathbf{C})\to\mathrm{P}^5(\mathbf{C})$ be the Veronese embedding of degree 2, namely, the map defined by*

$$\begin{aligned}&(\xi_0:\xi_1:\xi_2)\in\mathrm{P}^2(\mathbf{C})\\&\quad\to(\xi_0^2:\xi_1^2:\xi_2^2:\xi_0\xi_1:\xi_0\xi_2:\xi_1\xi_2)=(x_0:x_1:x_2:y_0:y_1:y_2)\in\mathrm{P}^5(\mathbf{C}),\end{aligned}$$

and let $p:\mathrm{P}^5(\mathbf{C})\to\mathrm{P}^3(\mathbf{C})$ be the linear projection defined by

$$\begin{aligned}&(x_0:x_1:x_2:y_0:y_1:y_2)\in\mathrm{P}^5(\mathbf{C})\\&\qquad\to(y_0:y_1:y_2:-(x_0+x_1+x_2))=(x:y:z:w)\in\mathrm{P}^3(\mathbf{C}).\end{aligned}$$

Then the hypersurface in $\mathrm{P}^3(\mathbf{C})$ defined by the equation $(xy)^2+(yz)^2+(zx)^2+xyzw=0$ coincides with $(p\circ v)(\mathrm{P}^2(\mathbf{C}))$, which is an algebraic surface with ordinary singularities, known as the Steiner surface.

The proof of this proposition is a direct calculation.

Theorem 1. *The normalization of the singularity defined by the equation (2.2) at the origin of $\mathbf{C}^4$ is a rational isolated singularity with multiplicity 4, and is rigid under deformations.*

Proof. We denote by S the Steiner surface, i.e., the projective variety in $\mathrm{P}^3(\mathbf{C})$ defined by the equation $(xy)^2+(yz)^2+(zx)^2+xyzw=0$, and by C_S the affine variety in $\mathbf{C}^4$ defined by the same equation, i.e., the cone over S. We denote by X the image of $\mathrm{P}^2(\mathbf{C})$ in $\mathrm{P}^5(\mathbf{C})$ by the Veronese embedding of degree 2, and by C_X the affine variety in $\mathbf{C}^6$ corresponding to X, i.e., the cone over X. Note that C_X is non-singular outside the origin of $\mathbf{C}^6$, since X is non-singular. We denote by $\overline{p}:\mathbf{C}^6\to\mathbf{C}^4$ the linear projection induced by $p:\mathrm{P}^5(\mathbf{C})\to\mathrm{P}^3(\mathbf{C})$ in Proposition 3. Since $S=p(X)$, we have $\overline{p}(C_X)=C_S$, We denote by $n:C_X\to C_S$ the restriction $\overline{p}$ to C_X. Since

$$\mathcal{O}_X(\nu):=\mathcal{O}_X([H_{\mathrm{P}^5(\mathbf{C})}]^{\otimes\nu})\simeq\mathcal{O}_{\mathrm{P}^2(\mathbf{C})}([H_{\mathrm{P}^2(\mathbf{C})}]^{\otimes 2\nu}),$$

$H^0(\mathrm{P}^5(\mathbf{C}),\mathcal{O}_{\mathrm{P}^5(\mathbf{C})}(\nu))\to H^0(X,\mathcal{O}_X(\nu))$ is surjective for every integer ν, where $[H_{\mathrm{P}^5(\mathbf{C})}]$ and $[H_{\mathrm{P}^2(\mathbf{C})}]$ denote the hyperplane line bundles on $\mathrm{P}^5(\mathbf{C})$ and $\mathrm{P}^2(\mathbf{C})$, respectively. Therefore X is *projectively normal*, and equivalently C_X is normal (cf. [2]). Hence $n:C_X\to C_S$ gives the normalization. To see that C_X has a rational isolated singularity, we take the blowing-up $\hat{b}:\widehat{\mathbf{C}^6}\to\mathbf{C}^6$ at the origin of $\mathbf{C}^6$. We put $\widehat{C_X}:=\hat{b}^{-1}(C_X)$, the proper inverse image of C_X by $\hat{b}$, and denote by $b : \widehat{C_X} \to C_X$ the restriction of $\hat{b}$ to $\widehat{C_X}$. Here we should remember that $\widehat{C^6}$ can be identified with $[H_{P^5(C)}]^{-1}$, $\widehat{C_X}$ with $[H_{P^5(C)}]^{-1}_{|X}$, the restriction of $[H_{P^5(C)}]^{-1}$ to X, and $b^{-1}(0)$ with the zero cross-section of the line bundle $L := [H_{P^5(C)}]^{-1}_{|X} \to X$. By these identifications, for any open neighborhood U of $b^{-1}(0)$ in $\widehat{C_X}$, we have

$$H^q(U, \mathcal{O}_U) \simeq \oplus_{\nu \geq o} H^q(X, L^{-\nu}) \simeq \oplus_{\nu \geq o} H^q(\mathrm{P}^2(\mathrm{C}), \mathcal{O}_{\mathrm{P}^2(\mathrm{C})}(2\nu)) = 0$$

for any $q \geq 1$. Hence $(R^q b_* \mathcal{O}_{\widehat{C_X}})_0 = 0$ for any $q \geq 1$, that is, $(C_X, 0)$ is a rational isolated singularity. The fact that C_X is rigid under deformatons follows from the following result due to M. Schlessinger ([2]):

Theorem The cone over a strongly rigid projective manifold is rigid under deformations.

Here, a projective manifold $Y \subset \mathrm{P}^n(\mathrm{C})$, $dim_{\mathrm{C}} Y > 0$, is defined to be *strongly rigid* if

(i) Y is projectively normal,
(ii) $H^1(Y, \Theta_Y(\nu)) = 0, -\infty < \nu < \infty$,
(iii) $H^1(Y, \mathcal{O}_Y(\nu)) = 0, -\infty < \nu < \infty$,

where Θ_Y denotes the sheaf of holomorphic vector fields on Y, and $F(\nu)$ denotes the sheaf F tensored with ν-th power of hyperplane line bundle.

Q.E.D.

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