Normalization of a certain degenerate ordinary triple point of dimension three *

Shoji Tsuboi

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1 The singularity $(xy)^2+(yz)^2+(zx)^2+xyzw=0$

We consider the following affine threefold:

$$\mathbb{C}^4 \supset T : f := (xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0 \tag{1}$$

where (x, y, z, w) denotes the coordinate on $\mathbb{C}^4$. As shown in [6], T has an ordinary triple point at $(0, 0, 0, w)$ if $w \neq 0$. Hence, we may think of the singularity $(T, 0)$ of T at the origin of $\mathbb{C}^4$ as a *degenerate ordinary triple point*. Let

$$P^3(\mathbb{C}) \supset S : f := (xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0 \tag{2}$$

be the hypersurface in $P^3(\mathbb{C})$, defined by the same polynomial f that defines T. This surface S is classically known as the *Steiner surface*. The surface S is obtained by projecting $P^2(\mathbb{C})$, embedded in $P^5(\mathbb{C})$ by monomials of degree 2, to $P^3(\mathbb{C})$. Indeed, if we denote by X the image of $P^2(\mathbb{C})$ in $P^5(\mathbb{C})$ by the map $\nu : P^2(\mathbb{C}) \to P^5(\mathbb{C})$ defined by

$$\begin{aligned}(\xi_0 : \xi_1 : \xi_2) \in P^2(\mathbb{C}) \quad \mapsto \quad & (\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_1\xi_2) \\ & = (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in P^5(\mathbb{C}),\end{aligned}$$

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then the surface S coincides with the image of X by the linear projection $p : P^5(\mathbb{C}) \to P^3(\mathbb{C})$ defined by

$$(x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in P^5(\mathbb{C}) \mapsto (y_0 : y_1 : y_2 : -(x_0 + x_1 + x_2)) \quad (3)$$
$$= (x : y : z : w) \in P^3(\mathbb{C}).$$

Applying the calculation in [6], we can see that S is an algebraic surface with *ordinary singularities*, whose singular locus D_S is $\{x = y = 0\} \cup \{y = z = 0\} \cup \{z = x = 0\}$, and that S has one ordinary triple point at $[0 : 0 : 0 : 1]$, six cuspidal points at $[0 : 0 : \pm 2 : 1], [0 : \pm 2 : 0 : 1], [\pm 2 : 0 : 0 : 1]$, and ordinary double points at other points of D_S. We denote by C_S the *cone* over S, which is nothing but T. We denote by C_X the cone over X. Since X is a non-singular, *projectively normal* subvariety in $P^5(\mathbb{C})$, whose degree is 4, $(C_X, 0)$ is a quadruple *normal* singular point (cf. [5] and [2, Exercise 3.4 (e), p.394]). Hence, if we denote by $\overline{p} : \mathbb{C}^6 \to \mathbb{C}^4$ the linear projection induced by $p : P^5(C) \to P^3(C)$ in (3), and by $n : C_X \to C_S$ the restriction of $\overline{p}$ to C_X, then $n : C_X \to C_S$ gives the normalization of $(T, 0) = (C_S, 0)$. We are interested in the normal singularity $(C_X, 0)$. It becomes non-singular after a single blowing-up. Indeed, as for the blowing-up $\hat{\tau} : \widehat{\mathbb{C}^6} \to \mathbb{C}^6$ at the origin of $\mathbb{C}^6$, $\widehat{\mathbb{C}^6}$ can be identified with $[H_{P^5(\mathbb{C})}]^{-1}$, where $[H_{P^5(\mathbb{C})}]$ denotes the line bundle determined by a hyperplane $H_{P^5(\mathbb{C})}$ of $P^5(\mathbb{C})$; the proper inverse image $\widehat{C_X}$ of C_X by τ with $[H_{P^5(\mathbb{C})}]^{-1}_{|X} \simeq [H_{P^2(\mathbb{C})}]^{-2}$, where $[H_{P^5(\mathbb{C})}]^{-1}_{|X}$ denotes the restriction of $[H_{P^5(\mathbb{C})}]^{-1}$ to X; and the exceptional divisor $E := \tau^{-1}(0)$ with the zero cross-section of the line bundle $L := [H_{P^2(\mathbb{C})}]^{-2}$ on $P^2(\mathbb{C})$. From this fact, it follows that $E^2 = -2H_{P^2(\mathbb{C})}$, where $H_{P^2(\mathbb{C})}$ denotes a hyperplane in $P^2(\mathbb{C})$.

Theorem 1.1 $(C_X, 0)$ *is*

(i) *rational, and so Cohen-Macaulay,*

(ii) *"rigid" under small deformations,*

(iii) *Gorenstein of index two,*

(iv) *terminal, and so canonical.*

Proof For the proofs of the assertions (i) and (ii), we refer to [6]. Here we give the proofs of the assertions (iii) and (iv).

(iii) First, we show that $(C_X, 0)$ is not 1-Gorenstein. We put

$$\begin{aligned} & I(C_X) \\ := & \ (x_0x_1 - y_0^2, x_1x_2 - y_2^2, x_0x_2 - y_1^2, x_1y_1 - y_0y_2, x_2y_0 - y_1y_2, x_0y_2 - y_0y_1), \end{aligned}$$

which is the ideal of C_X in the polynomial ring $\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$, and $R := \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]/I(C_X)$. We denote by A the localization R_m of R at $R - m$, where $m = (x_0, x_1, x_2, y_0, y_1, y_2)R$ is the maximal ideal corresponding to $0 \in C_X$, and by m_A the maximal ideal of the local ring A. Since x_0, x_1, x_2 is a regular sequence in R, so it is in A. We put $B := A/(x_0, x_1, x_2)A$, and denote by m_B the maximal ideal of B. Then we have

$$\mathrm{Ext}^3_A(\mathbb{C}, A) \simeq \mathrm{Hom}_B(\mathbb{C}, B) \qquad (\mathbb{C} \simeq A/m_A \simeq B/m_B). \tag{4}$$

(cf. [3, §18, Lemma 2]). Since

$$\begin{aligned} R/(x_0, x_1, x_2)R & \simeq \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]/\{I(C_X) + (x_0, x_1, x_2)\} \\ & \simeq \mathbb{C}[y_0, y_1, y_2]/(y_0^2, y_1^2, y_3^2, y_0y_2, y_1y_2, y_0y_1), \end{aligned}$$

if we put

$$\begin{aligned} R' & := \mathbb{C}[y_0, y_1, y_2]/(y_0^2, y_1^2, y_3^2, y_0y_2, y_1y_2, y_0y_1) \quad \text{and,} \\ m' & := (y_0, y_1, y_2)R', \end{aligned}$$

then we have $B \simeq R'_{m'}$. We denote by p, q and r the images in $R'_{m'}$ of y_0, y_1 and y_2, respectively, by the map $\mathbb{C}[y_0, y_1, y_2] \to R'_{m'}$. Then any element of $R'_{m'}$ can be represented by the linear combination of $1, p, q, r$ with coefficient in the complex number. For $\varphi \in \mathrm{Hom}_B(\mathbb{C}, B)$, we put

$$\varphi(1) = a_0 + a_1p + a_2q + a_3r \qquad (a_i \in \mathbb{C}, \ 0 \le i \le 3)$$

Since multiplying $1 \in \mathbb{C} \simeq B/mB$ by an element $b = b_0 + b_1p + b_2q + b_3r \in B$ as a B-module is given by $b \cdot 1 = b_0$, we have

$$\varphi(b \cdot 1) = \varphi(b_0) = b_0\varphi(1) = b_0(a_0 + a_1p + a_2q + a_3r). \tag{5}$$

On the other hand, we have

$$b\varphi(1) = b_0(a_0 + a_1p + a_2q + a_3r) + a_0(b_1p + b_2q + b_3r). \tag{6}$$

Since $\varphi(b \cdot 1) = b\varphi(1)$, comparing (6) and (5), we infer that $a_0 = 0$. Hence,

$$\varphi(1) = a_1 p + a_2 q + a_3 r \qquad (a_i \in \mathbb{C},\ 1 \leq i \leq 3).$$

Conversely, any element $\varphi \in \mathrm{Hom}_B(\mathbb{C}, B)$ is uniquely determined by this condition. Hence $\mathrm{Hom}_B(\mathbb{C}, B) \simeq \mathbb{C}^{\oplus 3}$, that is, by (4), $\mathrm{Ext}^3_A(\mathbb{C}, A) \not\simeq \mathbb{C}$. Therefore, the local ring A of C_X at 0 is not 1-Gorenstein (cf. [3, Theorem 18.1]).

Next, we show that $\omega^{[2]}_{C_X} := \mathcal{O}_{C_X}(2K_{C_X})$ is invertible, where ω_{C_X} and K_{C_X} denote the *canonical sheaf* and the *canonical divisor* (Weil divisor) of C_X, respectively. Note that

$$\mathbb{C}[x, y, z]^{\pm} \simeq \mathbb{C}[x^2, y^2, z^2, xy, zx, yz],$$

where $\mathbb{C}[x, y, z]^{\pm}$ denotes the subring of $\mathbb{C}[x, y, z]$ invariant under the transformation $(x, y, z) \to (-x, -y, -z)$, and also note that the kernel of the map

$$\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2] \to \mathbb{C}[x^2, y^2, z^2, xy, zx, yz]$$

defined by

$$x_0 \mapsto x^2, \quad x_1 \mapsto y^2, \quad x_2 \mapsto z^2, \quad y_0 \mapsto xy, \quad y_1 \mapsto zx, \quad y_2 \mapsto yz$$

is nothing but the homogeneous ideal of $X := \nu(P^2(\mathbb{C}))$ in $P^5(\mathbb{C})$, which is also the ideal $I(C_X)$ of C_X in $\mathbb{C}^6$. Therefore,

$$\mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]/I(C_X) \simeq \mathbb{C}[x^2, y^2, z^2, xy, yz, zx].$$

Hence, we conclude that $C_X \simeq \mathbb{C}^3/\{\pm 1\}$. From here on, we identify C_X with $\mathbb{C}^3/\{\pm 1\}$. We take out the invariant monomials

$$u_1 = x^2, \quad u_2 = xy, \quad u_3 = xz.$$

from coordinate functions on $\mathbb{C}^3/\{\pm 1\}$. We put

$$s = (dx \wedge dy \wedge dz)^2$$

Since $du_1 = 2x dx$, $du_2 = y dx + x dy$ and $du_3 = z dx + x dz$,

$$du_1 \wedge du_2 \wedge du_3 = 2x^3 dx \wedge dy \wedge dz.$$

Hence,

$$s = \frac{(du_1 \wedge du_2 \wedge du_3)^2}{4x^6} = \frac{1}{4}\frac{(du_1 \wedge du_2 \wedge du_3)^2}{u_1^3}.$$

This is a rational differential on C_X having no zeros and poles, and is thus a generator of $\omega_{C_X}^{[2]} := \mathcal{O}_{C_X}(2K_{C_X})$. Therefore, $(C_X, 0)$ is Gorenstein of index 2.
(iv) We denote by $\tau : \widehat{C}_X \to C_X$ the blowing-up of C_X at the origin. On one of affine pieces of $\widehat{C}_X$, we take ν_1, ν_2, ν_3 with

$$u_1 = \nu_1, \quad u_2 = \nu_1\nu_2, \quad u_3 = \nu_1\nu_3$$

as coordinates. Since

$$\tau^*(du_1 \wedge du_2 \wedge du_3) = \nu_1^2 d\nu_1 \wedge d\nu_2 \wedge d\nu_3,$$

we have

$$\tau^* s = \frac{1}{4}\nu_1(d\nu_1 \wedge d\nu_2 \wedge d\nu_3)^2$$

This means $\tau^*\omega_{C_X}^{[2]} = \mathcal{O}_{\widehat{C}_X}(-E + 2K_{\widehat{C}_X})$, where E denotes the exceptional divisor, that is,

$$K_{\widehat{C}_X} = \tau^* K_{C_X} + \frac{1}{2}E \quad \text{in} \quad Div(\widehat{C}_X) \otimes \mathbb{Q}.$$

Therefore, by definition, $(C_X, 0)$ is a terminal singularity.

2 An example of singular hypersurfaces in $P^4(\mathbb{C})$, which have degenerate ordinary triple points

Let H_i $(1 \le i \le 4)$ be non-singular hypersurfaces of degrees r_i $(1 \le i \le 4)$, respectively, in the complex projective 4-space $P^4(C)$ such that they are in general position at every point where they intersect. Let f_i $(1 \le i \le 4)$ be the homogeneous polynomial of degree r_i which defines the hypersurface H_i. We may assume $r_1 \ge r_2 \ge r_3 \ge r_4$ because of symmetry. We choose and fix a positive integer n with $n \ge 2r_1 + 2r_2 + 2r_3$. Let $\overline{X}$ be a hypersurface in $P^4(C)$ defined by the equation

$$F := Af_1f_2f_3f_4 + B(f_1f_2f_3)^2 + C(f_1f_2f_4)^2 + D(f_1f_3f_4)^2 + E(f_2f_3f_4)^2 = 0, \quad (7)$$

where A, B, C, D and E are homogeneous polynomials of five variables of respective degrees $n-r_1-r_2-r_3-r_4$, $n-2r_1-2r_2-2r_3$, $n-2r_1-2r_2-2r_4$, $n-2r_1-2r_3-2r_4$ and $n-2r_2-2r_3-2r_4$. We put $D^{(ij)}_{\overline{X}} := H_i \cap H_j$ $(1 \leq i < j \leq 4)$ and $D_{\overline{X}} := \bigcup_{1 \leq i<j \leq 4} D^{(ij)}_{\overline{X}}$. Then, by Bertini's theorem, $\overline{X}$ is non-singular outside $D_{\overline{X}}$ if we choose sufficiently generic A, B, C, D and E. In [6] we have shown the following proposition:

Proposition 2.1 *If the homogeneous polynomials* A, B, C, D *and* E *are chosen sufficiently generic, then* $\overline{X}$ *is locally isomorphic to one of the following germs of three dimensional hypersurface singularities at the origin of* $\mathbb{C}^4$ *at every point of* $\overline{X}$*:*

(i) $w = 0$ (*simple point*),
(ii) $zw = 0$ (*ordinary double point*),
(iii) $yzw = 0$ (*ordinary triple point*),
(iv) $xyzw = 0$ (*ordinary quadruple point*),
(v) $xy^2 - z^2 = 0$ (*cuspidal point*),
(vi) $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$ (*degenerate ordinary triple point*),

where (x, y, z, w) *is the coordinate on* $\mathbb{C}^4$.

We consider the following commutative diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{g_1} & Y_1 \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\sigma} \\ X & \xrightarrow{g} & Y, \end{array}$$

where $Y = P^4(\mathbb{C})$; $g : X \to Y$: the composite of the normalization map $n : X \to \overline{X}$ and the inclusion $\iota : \overline{X} \subset Y$; $\sigma : Y_1 \to Y$: the blowing-up of Y along the degenerate ordinary triple point locus $\sum \overline{d}$ of $\overline{X}$; $X_1 := X \times_Y Y_1$: the fiber product of X and Y_1 over Y; $\tau : X_1 \to X$: the projection to the first factor of $X_1 := X \times_Y Y_1$, which is nothing but the blowing-up of X along $\sum d$, the inverse image of $\sum \overline{d}$ by g; $g_1 : X_1 \to Y_1$: the projection to the second factor of $X_1 := X \times_Y Y_1$, which is nothing but the composite of the normalization map $n_1 : X_1 \to \overline{X}_1$ and the inclusion $\iota : \overline{X}_1 \subset Y_1$, where $\overline{X}_1$ denotes the proper inverse image of $\overline{X}$ by σ. Note that X is a normal algebraic variety which has isolated singular points described in §1 only; $\overline{X}_1$ is an algebraic variety which has the singular points (i), (ii), (iii), (iv) and (v) only; and X_1 is non-singular.

Lemma 2.1 *For a line bundle* $\mathbf{L}$ *on* Y_1 *and natural numbers* a, b, *we have the sheaf isomorphism*

$$g_{1*}(\mathcal{O}_{X_1}(g_1^*\mathbf{L}\otimes[-\sum_{d} aE_d - bD_{X_1}])) \simeq \mathcal{O}_{\overline{X}_1}(\mathbf{L}\otimes[\sum_{\overline{d}} E_{\overline{d}}]^{-a})\otimes_{\mathcal{O}_{Y_1}} \mathcal{I}(D_{\overline{X}_1})^b \quad (8)$$

where $E_d := \tau^{-1}(d)$ *for* $d \in \sum d := g^{-1}(\sum \overline{d})$; $E_{\overline{d}} := \sigma^{-1}(\overline{d})$ *for* $\overline{d} \in \sum \overline{d}$; $D_{\overline{X}_1}$ *the singular locus* $\overline{X}_1$; $D_{X_1} := g^{-1}(D_{\overline{X}_1})$; *and* $\mathcal{I}(D_{\overline{X}_1})$ *the ideal sheaf of* $D_{\overline{X}_1}$ *in* $\mathcal{O}_{Y_1}$.

Proof This can be proved by direct calculation, using the description of the map $g_1 : X_1 \to Y_1$ in termes of local coordinates. For example, if $p \in E_{\overline{d}}$ is a cuspidal point of $\overline{X}_1$, then the map $g_1 : X_1 \to Y_1$ is described as $(u, v, w) \mapsto (u^2, v, uv, w) = (x, y, z, w)$ at $g_1^{-1}(p)$, where $\overline{X}_1 : xy^2 - z^2 = 0$, $E_{\overline{d}} : w = 0$, locally, and $\mathcal{I}(D_{\overline{X}_1}) = (y, z)\mathcal{O}_{Y_1}$.

Proposition 2.2

$$\begin{aligned} H^p(X, \omega_X^{[2]}) &\simeq H^p(X_1, \mathcal{O}_{X_1}([2K_{X_1} - \sum_{d} E_d])) \\ &\simeq H^p(\overline{X}_1, \mathcal{O}_{\overline{X}_1}([2(n-5)H_1 - \sum_{\overline{d}} 3E_{\overline{d}}]) \otimes_{\mathcal{O}_{Y_1}} \mathcal{I}(D_{\overline{X}_1})^2) \qquad (p \geq 0) \end{aligned}$$

Proof By the *projection formula*,

$$\begin{aligned} R^q\tau_*\tau^*\omega_X^{[2]} &\simeq \omega_X^{[2]} \otimes_{\mathcal{O}_X} R^q\tau_*\mathcal{O}_{X_1} \\ &= \begin{cases} \omega_X^{[2]} & q = 0, \\ 0 & q \geq 1. \end{cases} \end{aligned}$$

Hence, by Leray's spectral sequence, we have

$$H^p(X, \omega_X^{[2]}) \simeq H^p(X_1, \tau^*\omega_X^{[2]}) \simeq H^p(X_1, \mathcal{O}_{X_1}([2K_{X_1} - \sum_{d} E_d])) \quad (9)$$

By the *double popine formula* (cf. [1, Theorem 9.3, p.166]),

$$K_{X_1} = g_1^*(\overline{X}_1 + K_{Y_1}) - D_{X_1} \quad (10)$$

where D_{X_1} := the closure of $\{ x \in X_1 \mid \#g_1^{-1}(g(x)) \geq 2 \}$. Since $\sigma^*[\overline{X}] = [\overline{X}_1] + 4\sum_{\overline{d}}[E_{\overline{d}}]$ and $K_{Y_1} = \sigma^*K_Y + 3\sum_{\overline{d}}[E_{\overline{d}}]$, where $\sum_{\overline{d}}$ denotes the sum in which $\overline{d}$ runs all over degenerate ordinary triple points of $\overline{X}$, we have

$$\overline{X}_1 + K_{Y_1} = \sigma^*[\overline{X} + K_Y] - \sum_{\overline{d}} [E_{\overline{d}}]. \quad (11)$$

Since $\overline{X}+K_Y = (n-5)H$, where H is a generic hyperplane in Y and $n = \deg\overline{X}$, if we denote by H_1 the proper image of H, by (10) and (11),

$$K_{X_1} = g_1^*((n-5)H_1) - \sum_d [E_d] - D_{X_1}.$$

Hence, $2K_{X_1} - \sum_d E_d = g_1^*(2(n-5)H_1) - \sum_d [3E_d] - 2D_{X_1}$. Therefore, by (9) and Lemma 2.1, we have the second isomorphism between cohomologies in the proposition.

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Shoji TSUBOI
Department of Mathematics and Computer Science
Faculty of Science
Kagoshima University
Korimoto 1-chome, 34-9
Kagoshima, 890-0065 JAPAN