Linear Projections of Rational Threefolds

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Abstract. In [17] we have proved formulas which give the Chern numbers of the normalization X of a hypersurfase with ordinary singularities $\overline{X}$ in $\mathsf{P}^4(\mathbf{C})$, though we have no concrete examples such hypersurafces whose structures are known. In this article we report some results of our trial to find out concrete examples of hypersurfases with ordinary singularities in $\mathsf{P}^4(\mathbf{C})$, which are expected to be useful to see that our formulas for the Chern numbers certainly hold. As a result, it turns out that the formulas for $\int_X c_3$ and $\int_X c_1^3$ in [17] are incorrect. We give the formulas amended correctly.

1 3-dimensional hypersurfaces with ordinary singularities

Throughout this article we work over the complex number field $\mathbf{C}$. We begin with recalling the definiton of 3-dimensional hypersurfaces with ordinary singularities.

Definition 1.1. ([13]) An irreducible hypersurface X in a 4-dimensional complex manifold Y is called a *3-dimensional hypersurface with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbf{C}^4$ at every point of X:

$$\begin{cases} (i)\ w=0\ \text{(simple point)} & (ii)\ zw=0\ \text{(ordinary double point)} \\ (iii)\ yzw=0\ \text{(ordinary triple point)} & (iv)\ xyzw=0\ \text{(ordinary quadruple point)} \\ (v)\ xy^2-z^2=0\ \text{(cuspidal point)} & (vi)\ w(xy^2-z^2)=0\ \text{(stationary point)}, \end{cases}$$

where (x, y, z, w) is the coordinate on $\mathbf{C}^4$.

Examples of 3-dimensional hypersurfaces with ordinary singularities are obtained by projecting smooth threefolds, i.e., smooth algebaric varieties of dimension 3, embedded in a complex projective space $\mathrm{P}^N(\mathbb{C})$ $(N \geq 5)$ to its 4-dimensional linear subspace by a *generic* linear projection ([10], [14]). In the subsequence we will explain what the word "generic" means and why this fact holds. Let X be an n-dimensional smooth subvariety of $\mathrm{P}^N(\mathbb{C})$, and Λ an $(N\text{-}m\text{-}1)$-dimensional linear subspace of $\mathrm{P}^N(\mathbb{C})$, Y an m-dimensional linear subspace of $\mathrm{P}^N(\mathbb{C})$ such that Λ and Y are situated in general position. We assume that $X \cap \Lambda = \emptyset$, and so

$$(N-m-1)+n<N \Longleftrightarrow n<m+1.$$

Definition 1.2. For X, Λ and Y as above, we define the *linear projection* $\pi_\Lambda : X \to Y$ of X from Λ to Y by

$$\pi_\Lambda(x) := L(x,\Lambda) \cap Y \qquad (x \in X),$$

where $L(x, \Lambda)$ denotes the $(N-m)$-dimensional linear subspace of $\mathrm{P}^N(\mathbb{C})$ generated by x and Λ.

We denote by $G(N\text{-}m\text{-}1, N)$ the Grassmann variety of $(N\text{-}m\text{-}1)$-linear subspaces of $P^N(\mathbb{C})$. We consider Λ as an element of $G(N\text{-}m\text{-}1, N)$ and vary it.

Next, we give the definition of a *locally stable holomorphic map*. Let $f : M \to N$ be a holomorphic map between complex manifolds, and S a finite subset of M. We denote by $f : (M, S) \to (N, f(S))$ the *multi-germ of a holomorphic map* f at S.

Definition 1.3. A multi-germ of a holomorphic map $f : (M, S) \to (N, f(S))$ is defined to be *stable* if any deformation (= parametrized unfolding) of it is *trivial*.

Definition 1.4. A holomorphic map between complex manifolds $f : M \to N$ is defined to be *locally stable* if for any finite subset S of M, the multi-germ of a holomorphic map $f : (M, S) \to (N, f(S))$ is stable.

With these notation and terminology, we have:

Theorem 1.1. ([10]) *Let X, Λ and Y be the same as in Definition 1.2. If (n, m) belongs to the so-called "nice range", then there exists a dense open subset U of $G(N\text{-}m\text{-}1, N)$ such that, for any $\Lambda \in U$, the linear projection $\pi_\Lambda : X \to Y$ of X from Λ to Y is a locally stable holomorphic map.*

Here we do not explain what *"nice range"* is, but we only mention that in the case $m = n + 1$, (n, m) belongs to the *"nice range"* if and only if $n \leq 14$. From this theorem we can derive the following:

Proposition 1.2. *Let X be a smooth algebraic threefold embedded in $P^N(\mathbb{C})$ $(N \geq 5)$, and Λ an $(N\text{-}5)$-dimensional linear subspace of $P^N(\mathbb{C})$, Y a 4-dimensional linear subspace of $P^N(\mathbb{C})$ such that Λ and Y are situated in general position. Then there exists a dense open subset U of $G(N\text{-}m\text{-}1, N)$ such that, for any $\Lambda \in U$, the image of X in Y by the linear projection $\pi_\Lambda : X \to Y$ from Λ to Y is a hypersurface with ordinary singularities in Y.*

Roughly speaking, the proof of this proposition proceeds as follows: First note that the pair of integers $(3, 4)$ surely belongs to the so-called "nice range". Generally, stable holomorphic map germs at a point are classified by the $\mathbb{C}$-algebra

$$R_f := \mathcal{O}_{X,p}/f^*\mathfrak{m}_{Y,f(p)} \qquad (\mathfrak{m}_{Y,f(p)} : \text{the maximal ideal of } \mathcal{O}_{Y,f(p)})$$

associated to $f : (X, p) \to (Y, f(p))$. In the case where $\dim X = 3$ and $\dim Y = 4$, $\mathbb{C}$-algebras associated to stable holomorphic germs at a point are only one of the following:

$$A_0 = \mathbb{C}[[x]]/(x), \quad A_1 = \mathbb{C}[[x]]/(x^2).$$

$R_f \simeq A_0$ is the case when f is non-degenerated at x, i.e., the Jacobian df of f has maximal rank at x. The *normal form* of the stable map germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ with $R_f \simeq A_1$ is given by

$$\begin{cases} y_1 \circ f = x_1 \\ y_2 \circ f = x_2 \\ y_3 \circ f = x_3^2 \\ y_4 \circ f = x_1 x_3, \end{cases}$$

and if we define

$$C(A_1) := \left\{ x \in \mathbb{C}^3 \mid R_{f_x} \simeq A_1 \right\}$$

where f_x denotes the map germ of f at $x \in \mathbb{C}^3$, then

$$C(A_1) : x_1 = x_3 = 0.$$

The equation of $f(\mathbb{C}^3) \subset \mathbb{C}^4$ at 0 is given by

$$y_3 y_1^2 - y_4^2 = 0,$$

which is the so-called *Whitney umbrella*, or *cuspidal point*, or *pinch point*. By this and the fact that a locally satable holomorphic map is a *Thom-Boardman map satisfying condition NC* (*normal crossing*), we have the proposition above (For details, see [14]). For the precise definition of a *Thom-Boardman map satisfying condition NC* (*normal crossing*), see [4].

2 Chern numbers of the normalization of a hypersurface with ordinary singularities in $\mathrm{P}^4(\mathbb{C})$

Throughout §§2, 3, we fix the notation as follows:

$Y := P^4(\mathbf{C})$: the complex projective 4-space,
$\overline{X}$: an algebraic threefold with ordinary singularities in Y,
$\overline{J}$: the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$,
$\overline{D}$: the singular locus of $\overline{X}$,
$\overline{T}$: the triple point locus of $\overline{X}$, which is equal to the singular locus of $\overline{D}$,
$\overline{C}$: the cuspidal point locus of $\overline{X}$, precisely, its closure, since we always consider $\overline{C}$ contains the stationary points,
$\Sigma\overline{q}$: the quadruple point locus of $\overline{X}$,
$\Sigma\overline{s}$: the stationary point locus of $\overline{X}$,
$n_{\overline{X}} : X \to \overline{X}$: the normalization of $\overline{X}$,
$f : X \to Y$: the composite of the normaliztion map $n_{\overline{X}}$ and the inclusion $\overline{\iota} : \overline{X} \hookrightarrow Y$,
J : the scheme-theoretic inverse of $\overline{J}$ by f,
D, T, C and Σs : the inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma\overline{s}$ by f, respectively.

We put

$$n := \deg \overline{X} \text{ (the degree of } \overline{X} \text{ in } P^4(C)),\ m := \deg \overline{D},\ t := \deg \overline{T},\ \gamma := \deg \overline{C}.$$

Note that $\overline{T}$ and $\overline{C}$ are smooth curves, intersecting transversely at $\Sigma\overline{s}$, and that the normalization X of $\overline{X}$ is also smooth. Calculating by use of local coordinates, we can easily see the following:

(i) J contains D, and the *residual scheme* to D in J is the reduced scheme C, i.e., $\mathcal{I}_J = \mathcal{I}_D \otimes_{\mathcal{I}_X} \mathcal{I}_C$, where $\mathcal{I}_J$, $\mathcal{I}_D$, $\mathcal{I}_C$ are the ideal sheaves of J, D and C, respectively (cf. [3], Definition 9.2.1, p.160);

(ii) D is a surface with ordinary singularities, whose singular locus is T,

(iii) D is the *double point* locus of the map $f : X \to Y$, i.e., the closure of $\{q \in X \mid \# f^{-1}(f(q)) \geq 2\}$;

(iv) the map $f_{|D} : D \to \overline{D}$ is generically two to one, simply ramified at C;

(v) the map $f_{|T} : T \to \overline{T}$ is generically three to one, simply ramified at Σs.

Concerning the Euler number of X, denoted by $\chi(X)$, we have the following:

Proposition 2.1. ([16], Proposition 2.3)

$$\chi(X) \;=\; n(2n^2-7n+9)-2(3n-7)m+6t-4\gamma-c \tag{2.1}$$

where c *denotes the class of* $\overline{X}$*, i.e., the number of hyperplanes being tangent to* X *at a point and passing through a fixed generic 2-linear subspace of* $P^4(\mathbf{C})$.

To prove the proposition above we use a Lefschetz pencil $\overline{\mathcal{L}} = \bigcup_{\lambda\in P^1}\overline{X}_\lambda$ on $\overline{X}$, consisting of hyperplane sections of $\overline{X}$. We denote by $\overline{B}$ the base point locus of $\overline{\mathcal{L}}$, which is an irreducible curve of degree n with m nodes on $\overline{X}$. Let $\sigma : \widetilde{X} \to X$ be the blowing-up along $n_{\overline{X}}^{-1}(\overline{B})$, and let $\widetilde{\mathcal{L}} = \bigcup_{\lambda\in P}\widetilde{X}_\lambda$ be the pull-back of $\overline{\mathcal{L}}$ to $\tilde{X}$ by $n_{\overline{X}}\circ\sigma$. Then $\widetilde{\mathcal{L}}$ gives a fibering of $\widetilde{X}$, whose fiber is a smooth surface except over finite points $\lambda_1,\cdots,\lambda_c$ of P^1. Every singular fiber over λ_i $(1\le i\le c)$ is a surface with only one isolated ordinary double point. The Euler number of a general fiber $\widetilde{X}_\lambda$ is given by

$$\chi(\widetilde{X}_\lambda) = n(n^2-4n+6)-(3n-8)m+3t-2\gamma,$$

wnich is a classical formula for surfaces with ordinary sungularites. From thses facts, (2.1) follows.

The formulas for the Chern numbers of X are as follows:

Theorem 2.2.

$$\int_X c_3 = \chi(X) = -n(n^3-5n^2+10n-10)+(4n^2-15n-2m+20)m-(4n-15)t$$
$$+(n+10)\gamma+5\deg[K_X\cdot C]-\#\Sigma\overline{s}+2\chi(\overline{C},\mathcal{O}_{\overline{C}})+4\#\Sigma\overline{q}.$$

$$\int_X c_1^3 = -n(n-5)^3+6(n-5)^2m-3(n-5)(nm+3t-\gamma)$$
$$+(n^2-2m)m+5nt-(2n-5)\gamma+\deg[K_X\cdot C]-\#\Sigma\overline{s}+4\#\Sigma\overline{q}.$$

$$\int_X c_1c_2 = -24\chi(X,K_X) = -24\,\chi(Y,\mathcal{O}_Y([(n-5)H]-\overline{D}))+24$$
$$= -(n-4)(n-3)(n-2)(n-1)+24\,\chi(\overline{D},\mathcal{O}_{\overline{D}}(n-5))+24,$$

where K_X *is a canonical divisor of* X.

Remark 2.1. The formulas for $\int_X c_3$ and $\int_X c_1^3$ in [17] are false. This is because the diagram

$$\begin{array}{ccc} C & \xrightarrow{\iota} & X \\ {\scriptstyle f_{|C}}\downarrow & & \downarrow{\scriptstyle f} \\ \overline{C} & \xrightarrow[\overline{\iota}]{} & Y. \end{array}$$

is not Cartesian, since $[f^{-1}(\overline{C})] = 2[C]$, and so we cannot apply the *excess intersection formula* (cf. [3], Theorem 6.3, p.102) to calculate $f^*[\overline{C}]$. Hence, the identity

$$f^*[\overline{C}] = f^*[\overline{X}]\cdot[C]-[D\cdot C],$$

on page 299 in [17] is incorrect, and the second identity at (3.26) on the same page in [17] must be replaced by

$$[D\cdot C] = f^*[\overline{X}+K_Y]\cdot[C]-[K_X\cdot C],$$

which follows from the *double point formula* $[D] = f^*[\overline{X}+K_Y]-[K_Y]$, where K_X and K_Y are canonical divisors of X and Y, respectively.

The most hard part of Theroem 2.2 is the first formula for the Euler number. The class c is nothing but the degree of the top polar class of $\overline{X}$. Thanks to Piene's formula in [11], calculating the Segre classes of the singular subscheme $\overline{J}$ of $\overline{X}$, we have

$$c = (n-1)^3n-(4n^2-9n-2m+6)m+(4n-9)t-(n+14)\gamma-5\deg[K_X\cdot C]+\#\Sigma\overline{s}-2\chi(\overline{C},\mathcal{O}_{\overline{C}})-4\#\Sigma\overline{q}.$$

For the precise definitions of *polar class* and *Segre class*, see [11] or [3].

3 Linear projections of rational threefolds

The following is an example of a 2-dimensional hypersurafface with ordinary singularities in $\mathsf{P}^3(\mathbb{C})$, named *Steiner surface*:

$$(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0,$$

where $[x : y : z : w]$ is the homogeneous coordinate on $\mathsf{P}^3(\mathbb{C})$. Its singular locus consists of the three lines Λ_0, Λ_1 and Λ_2 defined by $x = y = 0$, $y = z = 0$ and $z = x = 0$, respectively, which we call the *double curves* of it. The Steiner surface has one ordinary triple point $[0 : 0 : 0 : 1]$, six ordinary cuspidal points $[1 : 0 : 0 : \sqrt{2}]$, $[1 : 0 : 0 : -\sqrt{2}]$, $[0 : 1 : 0 : \sqrt{2}]$, $[0 : 1 : 0 : -\sqrt{2}]$, $[0 : 0 : 1 : \sqrt{2}]$, $[0 : 0 : 1 : -\sqrt{2}]$, two of which lie on each of the line Λ_i, and no quadruple point. The Steiner surface is obtained as the image of the composite of the quadratic Veronese map (embedding)

$$\begin{array}{rcl} \nu_2 : [\xi_0 : \xi_1 : \xi_2] & \in & \mathrm{P}^2(\mathbf{C}) \\ & \to & [\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_1\xi_2] = [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \in \mathrm{P}^5(\mathbf{C}) \end{array}$$

and the linear projection

$$\begin{array}{rcl} \pi_L : (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) & \in & \mathrm{P}^5(\mathbf{C}) \\ & \to & (y_0 : y_1 : y_2 : -(x_0 + x_1 + x_2)) = (x : y : z : w) \in \mathrm{P}^3(\mathbf{C}) \end{array}$$

The center of the linear projection π_L is the line

$$L : y_0 = y_1 = y_2 = x_0 + x_1 + x_2 = 0.$$

In what follows we try to find out similar examples in 3-dimensional case. First, we recall the formulas which express *multiple-point cycle classes* and *ramification point cycle classes* in terms of invariants of X and Y for an appropriately generic morphism $f : X \to Y$ between smooth algebraic varieties with $\dim X < \dim Y$, where *cycle classes* mean equivalence classes in the ring $A.X$ of algebraic cycles on X modulo rational equivalence. We set

$$M_r := \Big\{ x \in X \mid \text{there exist } r \text{ distinct points (possibly infinitely near each other) in } f^{-1}f(x) \Big\},$$

and call it the r-*fold point locus of* f. M_r has naturally the structure of a reduced subscheme. We denote by $[M_r]$ the element of $A.X$ determined by M_r. We set $n = \dim X$, $m = \dim Y$ $(n < m)$, and

$$R := \Big\{ x \in X \mid \mathrm{rank}(df)_x \leq n - 1 \Big\},$$

where df is the Jacobian map of f. R is called the *ramification locus* of f, or the *singular locus* of f. R has naturally subscheme structure; it is defined by the ideal generated by the n-minors of $df : \tau_X \to f^*\tau_Y$, where τ_X and τ_Y denote the tangent bundles of X and Y, respectively. We denote by $[R]$ the element of $A.X$ determined by R.

Theorem 3.1. *Let X be a smooth algebraic threefold embedded in $\mathrm{P}^N(\mathbf{C})$ $(N \geq 5)$, Y a 4-dimensional linear subspace of $\mathrm{P}^N(\mathbf{C})$, and $\pi_\Lambda : X \to Y$ the linear projection of X from an $(N-5)$-dimensional linear subspace Λ of $\mathrm{P}^N(\mathbf{C})$ to Y. We denote by $\overline{X}$ the image of X by π_Λ. If π_Λ is generic, that is, if Λ corresponds to a point of a suitable dense open subset of the Grassmann varity $G(N\text{-}5, N)$, then M_i is empty for $i \geq 5$ and*

$$\dim M_i = 4 - i \qquad (2 \leq i \leq 4).$$

Furthermore, under the same assumption, we have:

$$[M_2] = \pi_\Lambda^*[\overline{X} + K_Y] - K_X,$$

$$[M_3] = \frac{1}{2!}\Big\{ [M_2]^2 - [M_2] \cdot \pi_\Lambda^* c_1(Y) + 2c_2(\nu) + \pi_\Lambda^*\pi_{\Lambda *}c_1(X) - c_1(\nu)c_1(X) \Big\},$$

$$[M_4] = \frac{1}{3!}\Big\{ \pi_\Lambda^*\pi_{\Lambda *}2![M_3] - 3c_1(\nu) \cdot (2![M_3]) + 6c_2(\nu)[M_2] - 6c_1(\nu)c_2(\nu) - 12c_3(\nu) \Big\},$$

where $\nu := \pi_\Lambda^*\tau_Y - \tau_X$ is an element of $K(X)$, called the virtual normal sheaf of π_Λ.

The above theorem is a conclusion derived from *multiple-point formulas* due to S. L. Kleimen ([6], [7]).

Theorem 3.2. *With the same notation and under the same assumption as in Theorem 3.1,* R *is a smooth curve (possibly reducible), and*

$$[R] = c_2(\nu).$$

The fact that R is smooth follows from that π_Λ is a Thom-Boardman map. Tha last identity in the theorem is a conclusion derived from the *Porteous formula* ([12]).

In the subsequence, we denote by H_{P^n} a generic hyperplane in $P^n(\mathbf{C})$, and by $H^i_{P^n}$ the intersection of i hyperplanes in general position in $P^n(\mathbf{C})$.

Example 3.1 (Generic projection of Segre threefold): Let $s : P^1(\mathbf{C}) \times P^2(\mathbf{C}) \to P^5(\mathbf{C})$ be the map defined by

$$\begin{aligned}[s_0 : s_1] \times [t_0 : t_1 : t_2] &\in P^1(\mathbf{C}) \times P^2(\mathbf{C}) \\ &\to [s_0t_0 : s_0t_1 : s_0t_2 : s_1t_0 : s_1t_1 : s_1t_2] = [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \in P^5(\mathbf{C})\end{aligned}$$

i.e., the *Segre map* from $P^1(\mathbf{C}) \times P^2(\mathbf{C})$ to $P^5(\mathbf{C})$. We set

$$\Sigma_{1,2} := s(P^1(\mathbf{C}) \times P^2(\mathbf{C})),$$

which is called *Segre threefold*. It is a *rational normal scroll*, and as such is denoted by $X_{1,1,1}$, whose meaning is as follows: We take three points p_0, p_1, p_2 in general position in $P^2(\mathbf{C})$,and set

$$\begin{aligned}L_0 &:= s(P^1(\mathbf{C}) \times p_0),\\ L_1 &:= s(P^1(\mathbf{C}) \times p_1)\\ L_2 &:= s(P^1(\mathbf{C}) \times p_1).\end{aligned}$$

These are three lines in general position in $P^5(\mathbf{C})$. We denote the natural isomorphisms

$$\varphi_i : L_0 \to L_i \quad (i = 1, 2).$$

Then $\Sigma_{1,2}$ is described as

$$\Sigma_{1,2} = \bigcup_{p \in L_0} \overline{p, \varphi_1(p), \varphi_2(p)},$$

where $\overline{p, \varphi_1(p), \varphi_2(p)}$ denotes the 2-dimensional linear subspace of $P^5(\mathbf{C})$, generated by $p, \varphi_1(p)$ and $\varphi_2(p)$.

Proposition 3.3. *We denote by* $\overline{\Sigma_{1,2}}$ *the image of* $\Sigma_{1,2}$ *by a generic linear projection from a point* $p \in P^5(\mathbf{C})$ *to* $P^4(\mathbf{C})$. *Then:*

$$\deg \overline{\Sigma_{1,2}} = 3.$$

Proof: By the definition of $s : P^1(\mathbf{C}) \times P^2(\mathbf{C}) \to P^5(\mathbf{C})$,

$$s^*[\Sigma_{1,2} \cap H_{P^5}] = [H_{P^1} \times P^2] + [P^1 \times H_{P^2}].$$

Hence

$$s^*[\Sigma_{1,2} \cap H^3_{P^5}] = ([H_{P^1} \times P^2] + [P^1 \times H_{P^2}])^3 = 3[H_{P^1} \times H^2_{P^2}].$$

Since $H_{P^1} \times H^2_{P^2}$ is a point of $P^1(\mathbf{C}) \times P^2(\mathbf{C})$,

$$\int_{P^4} \overline{\Sigma_{1,2}} \cap H^3_{P^4} = \int_{P^5} \Sigma_{1,2} \cap H^3_{P^5} = \int_{P^1 \times P^2} s^*[\Sigma_{1,2} \cap H^3_{P^5}] = 3,$$

i.e., $\deg \overline{\Sigma_{1,2}} = 3$. ∎

By Theorem 3.1 and Theorem 3.2, we have:

Proposition 3.4. *We denote by* $f : \mathrm{P}^1(\mathbf{C}) \times \mathrm{P}^2(\mathbf{C}) \to \mathrm{P}^4(\mathbf{C})$ *the composite of the Segre map* $s : \mathrm{P}^1(\mathbf{C}) \times \mathrm{P}^2(\mathbf{C}) \to \mathrm{P}^5(\mathbf{C})$ *and a generic linear projection* $\pi_p : \mathrm{P}^5(\mathbf{C}) \to \mathrm{P}^4(\mathbf{C})$*. Concerning the multiple-point loci and the singular locus of* f*, we have the following:*

$$[M_2] = [P^1 \times H_{P^2}] \tag{3.1}$$

$$[M_3] = [M_4] = 0, \tag{3.2}$$

$$[R] = [H_{P^1} \times H_{P^2}] + [P^1 \times H_{P^2}^2] \tag{3.3}$$

Proposition 3.5. *Concerning the various singular loci of* $\overline{X} := \overline{\Sigma}_{1,2} = f(P^1 \times P^2)$*, we have the following:*

$$\deg[\overline{D}] = 1, \tag{3.4}$$

$$\deg[\overline{C}] = 2, \tag{3.5}$$

$$[\overline{T}] = [\Sigma\overline{q}] = [\Sigma\overline{s}] = 0. \tag{3.6}$$

Proof: Since $f_*[M_2] = 2[\overline{D}]$, by the projection formula,

$$f_*([M_2] \cdot f^*[H_{P^4}^2]) = 2[\overline{D}] \cdot [H_{P^4}^2]. \tag{3.7}$$

Since

$$f^*[H_{P^4}] = [H_{P^1} \times P^2] + [P^1 \times H_{P^2}], \tag{3.8}$$

we have

$$f^*[H_{P^4}]^2 = ([H_{P^1} \times P^2] + [P^1 \times H_{P^2}])^2 = 2[H_{P^1} \times H_{P^2}] + [P^1 \times H_{P^2}^2].$$

Hence, by (3.1)

$$\begin{aligned} [M_2] \cdot f^*[H_{P^4}]^2 &= [P^1 \times H_{P^2}] \cdot (2[H_{P^1} \times H_{P^2}] + [P^1 \times H_{P^2}^2]) \\ &= 2[H_{P^1} \times H_{P^2}^2]. \end{aligned}$$

Therefore, since $H_{P^1} \times H_{P^2}^2$ is a point of $P^1 \times P^2$, by (3.7) we have

$$\int_{P^4} [\overline{D}] \cdot [H_{P^4}]^2 = 1.$$

Similarly, using the fact $f_*[R] = [\overline{C}]$, we can prove (3.5) as follows: By the projection formula,

$$f_*([R] \cdot f^*[H_{P^4}]) = [\overline{C}] \cdot [H_{P^4}]. \tag{3.9}$$

By (3.3) and (3.8),

$$\begin{aligned} [R] \cdot f^*[H_{P^4}] &= ([H_{P^1} \times H_{P^2}] + [P^1 \times H_{P^2}^2]) \cdot ([H_{P^1} \times P^2] + [P^1 \times H_{P^2}]) \\ &= [H_{P^1} \times H_{P^2}^2] + [H_{P^1} \times H_{P^2}^2] = 2[H_{P^1} \times H_{P^2}^2] \end{aligned}$$

Therefore, since $H_{P^1} \times H_{P^2}^2$ is a point of $P^1 \times P^2$, by (3.9),

$$\int_{P^4} [\overline{C}] \cdot [H_{P^4}] = \int_{P^1 \times P^2} [R] \cdot f^*[H_{P^4}] = 2.$$

■

By Proposition 3.3, Proposition 3.5, Proposition 2.1 and the formula for $\int_X c_3$ in Theorem 2.2, we have:

Proposition 3.6. *Concerning the class* c *of* $\overline{X}$ *and the Euler Poincaré characteristic* $\chi(\overline{C}, \mathcal{O}_{\overline{C}})$ *of the cuspidal point locus (smooth curve)* $\overline{C}$ *of* $\overline{X}$*, we have the following:*

$$c = 0, \qquad \chi(\overline{C}, \mathcal{O}_{\overline{C}}) = 1.$$

The concrete equation of $\overline{\Sigma_{1,2}}$ can be calculated as follows: The Gröbner basis of the homogeneous ideal of $\Sigma_{1,2}$ in $P^5(\mathbf{C})$ is given by

$$x_0y_1 - x_1y_0, \quad x_0y_2 - x_2y_0, \quad x_1y_2 - x_2y_1.$$

Hence the point $p := [1 : 0 : 0 : 0 : 1 : 0]$ is not included in $\Sigma_{1,2}$. We consider the projection π_p from the point p to the hyperplane

$$H : x_0 = 0.$$

This projection π_p is given by

$$\begin{aligned} [x_0 : x_1 : x_2 : y_0 : y_1 : y_2] \quad &\in \quad P^5(\mathbf{C}) \\ &\to (a|x)p - (a|p)x = x_0p - x = [0 : x_1 : x_2 : y_0 : y_1 - x_0 : y_2] \in H. \end{aligned}$$

where $a = [1 : 0 : 0 : 0 : 0 : 0]$ is the normal vector of the hypersurface H, and $(\ |\ )$ denotes the inner product. We regard H as $P^4(\mathbf{C})$ and denote its homogeneous coordinates by $[z_0 : z_1 : z_2 : z_3 : z_4]$. Then $\pi_p \circ s : P^1(\mathbf{C}) \times P^2(\mathbf{C}) \to P^4(\mathbf{C})$ is given by

$$\begin{aligned} [s_0 : s_1] \times [t_0 : t_1 : t_2] \quad &\in \quad P^1(\mathbf{C}) \times P^2(\mathbf{C}) \\ &\to [z_0 : z_1 : z_2 : z_3 : z_4] = [s_0t_1 : s_0t_2 : s_1t_0 : s_1t_1 - s_0t_0 : s_1t_2] \in P^4(\mathbf{C}). \end{aligned}$$

We set

$$\overline{X} := (\pi_p \circ s)(P^1(\mathbf{C}) \times P^2(\mathbf{C})).$$

Computing the Gröbner basis of the ideal of $\overline{X}$ in $P^4(\mathbf{C})$ by the aid of computer, we obtain the defining equation of $\overline{X}$ as follows:

$$\overline{X} : \ z_2z_1^2 + z_3(z_1z_4) - z_0z_4^2 = 0.$$

The singular loci of $\overline{X}$ are:

$$\begin{aligned} \overline{D} : &\quad \Big\{ z_1 = z_4 = 0 \Big\}, \\ \overline{C} : &\quad \Big\{ z_1 = z_4 = 0 \Big\} \cap \Big\{ z_3^2 + 4z_0z_2 = 0 \Big\}. \end{aligned}$$

Example 3.2 (Generic projection of rational scroll $X_{2,2,2}$ in $P^8(\mathbf{C})$): Let $\nu_2 : P^1(\mathbf{C}) \to P^2(\mathbf{C})$ be the quadratic Veronese map (embedding), $s : P^2(\mathbf{C}) \times P^2(\mathbf{C}) \to P^8(\mathbf{C})$ the Segre map, and consider the composition

$$P^1(\mathbf{C}) \times P^2(\mathbf{C}) \xrightarrow{\nu_2 \times \mathrm{id}} P^2(\mathbf{C}) \times P^2(\mathbf{C}) \xrightarrow{\ \ s\ \ } P^8(\mathbf{C}).$$

The image of this map is a *rational normal scroll*, and is denoted by $X_{2,2,2}$, whose meaning is as follows: We take three points p_0, p_1, p_2 in general position in the second factor $P^2(\mathbf{C})$, and set

$$\begin{aligned} L_0 &:= s(P^2(\mathbf{C}) \times p_0), \\ L_1 &:= s(P^2(\mathbf{C}) \times p_1), \\ L_2 &:= s(P^2(\mathbf{C}) \times p_2). \end{aligned}$$

These are three 2-dimensional linear subspaces in general position in $P^8(\mathbf{C})$. Furthermore, we set

$$C_0 := (s \circ (\nu_2 \times \mathrm{id}))(P^1(\mathbf{C}) \times p_0),$$

$$
\begin{aligned}
C_1 &:= (s \circ (\nu_2 \times \mathrm{id}))(\mathrm{P}^1(\mathbf{C}) \times p_1), \\
C_2 &:= (s \circ (\nu_2 \times \mathrm{id}))(\mathrm{P}^1(\mathbf{C}) \times p_2).
\end{aligned}
$$

Each C_i is a quadric in L_i. We denote the natural isomorphisms by

$$\varphi_i : C_0 \to C_i \quad (i = 1, 2).$$

Then $X_{2,2,2}$ is described as

$$X_{2,2,2} = \bigcup_{p \in C_0} \overline{p, \varphi_1(p), \varphi_2(p)},$$

where $\overline{p, \varphi_1(p), \varphi_2(p)}$ denotes the 2-dimensional linear subspace of $\mathrm{P}^8(\mathbf{C})$, generated by $p, \varphi_1(p)$ and $\varphi_2(p)$. We denote by $\overline{X_{2,2,2}}$ the image of $X_{2,2,2}$ by a *generic linear projection* to a 4-dimensional linear subspace of $\mathrm{P}^8(\mathbf{C})$. The center of this projection is a 3-dimensional linear subspace of $\mathrm{P}^8(\mathbf{C})$. By Theorem 3.1, Theorem 3.2, Proposition 2.1, the formula for $\int_X c_3$ in Theorem 2.2 and Remark 3.1 below, we have the following concerning the degrees of $\overline{X_{2,2,2}}$ itself and the various singular loci of it:

Proposition 3.7.

$$
\begin{aligned}
&\deg[\overline{X_{2,2,2}}] = 6, \quad \deg[\overline{D}] = 10, \quad \deg[\overline{T}] = 4, \quad \deg[\overline{C}] = 8, \quad \#[\Sigma\overline{q}] = 0, \quad \#[\Sigma\overline{s}] = 12, \\
&c = 0, \quad \chi(\overline{C}, \mathcal{O}_{\overline{C}}) = 1
\end{aligned}
$$

Example 3.3 (Steiner threefold): Let $\nu_2 : \mathrm{P}^3(\mathbf{C}) \to \mathrm{P}^9(\mathbf{C})$ be the map defined by

$$
\begin{aligned}
[\xi_0 : \xi_1 : \xi_2 : \xi_3] \quad \in \quad & \mathrm{P}^3(\mathbf{C}) \\
& \to [\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_3^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_0\xi_3 : \xi_1\xi_2 : \xi_1\xi_3 : \xi_2\xi_3] \\
& \quad = [x_0 : x_1 : x_2 : x_3 : y_0 : y_1 : y_2 : y_3 : y_4 : y_5] \in \mathrm{P}^9(\mathbf{C}),
\end{aligned}
$$

i.e., the quadratic Veronese map (embedding), Λ a 4-dimensional linear subspace of $\mathrm{P}^9(\mathbf{C})$, and π_Λ the linear projection from Λ to a 4-dimensional linear subsapce Y of $\mathrm{P}^9(\mathbf{C})$, situated in twisted position with respect to Λ. We set

$$X := \nu_2(\mathrm{P}^3(\mathbf{C})), \quad \overline{X} := \pi_\Lambda(X).$$

Let us call this $\overline{X}$ *Steiner threefold*. By the same way to prove Proposition 3.7, we have the following concerning the degrees of the Steiner threefold itself and various singular loci of it:

Proposition 3.8.

$$
\begin{aligned}
&\deg[\overline{X}] = 8, \quad \deg[\overline{D}] = 20, \quad \deg[\overline{T}] = 20, \quad \deg[\overline{C}] = 20, \quad \#[\Sigma\overline{q}] = 5, \quad \#[\Sigma\overline{s}] = 40, \\
&c = 4, \quad \chi(\overline{X}, \mathcal{O}_{\overline{X}}) = -10
\end{aligned}
$$

Remark 3.1. The number of stationary points $\Sigma\overline{s}$ in Proposition 3.7 and Proposition 3.8 can be calculated by the identity

$$f^*[\overline{T}] = f^*[\overline{X}] \cdot T - f^*[\overline{T}] - [\Sigma s] + [\Sigma q]$$

in Proposition 1.12 in [17].

It is difficult to calculate the concrete equations for $\overline{X_{2,2,2}}$ in Example 3.2 and $\overline{X}$ in Example 3.3 even if we use computer, though it sometimes happens that we obtain the concrete equations of 3-dimensional hypersurfaces in $\mathrm{P}^4(\mathbf{C})$, which have other kinds of multiple-points than *ordinary* and *stationary* quadruple points.

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