Infinitesimal Parameter Spaces of Locally Trivial Deformations of Compact Complex Surfaces with Ordinary Singularities

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Abstract

In this paper we shall give a description of the cohomology $H^1(S, \Theta_S)$ for a compact complex surface S with *ordinary singularities*, using a 2-cubic hyper-resolution of S in the sense of F. Guillén, V. Navarro Aznar *et al.* ([2]), where Θ_S denotes the sheaf of germs of holomorphic tangent vector fields on S. As a by-product, we shall show that the natural homomorphism $H^1(S, \Theta_S) \to H^1(X, \Theta_X(-log\, D_X))$ is injective under some condition, where X is the (non-singular) normal model of S, D_X the inverse image of the double curve D_S of S by the normalization map $f : X \to S$, and $\Theta_X(-log\, D_X)$ the sheaf of germs of logarithmic tangent vector fields along D_X on X.

1 2-cubic hyper-resolutions of compact complex surfaces with ordinary singularities

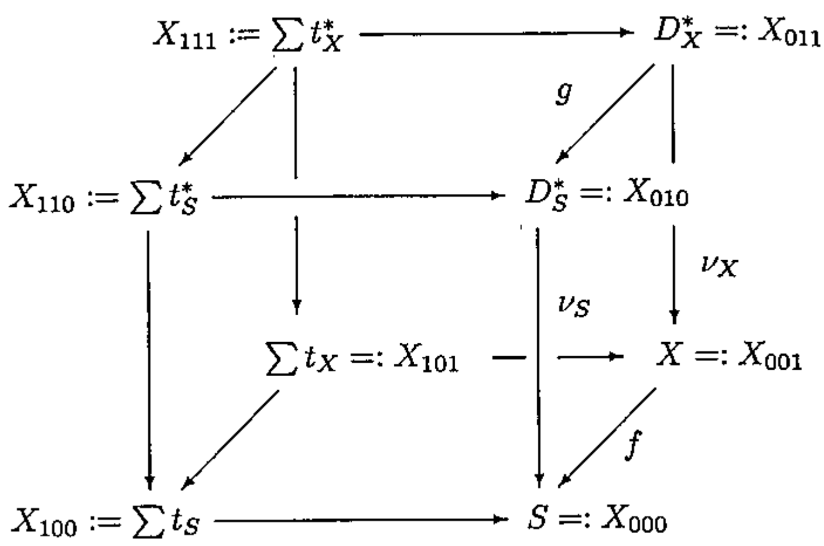
A 2-dimensional compact complex space S is called a compact complex surface with *ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurfaces at the origin of the complex 3-space C^3 at every point of S :

$$\begin{cases} (i)\ z = 0 \ \text{(simple point)} & (ii)\ yz = 0 \ \text{(ordinary double point)} \\ (iii)\ xyz = 0 \ \text{(ordinary triple point)} & (iv)\ xy^2 - z^2 = 0 \ \text{(cuspidal point)}, \end{cases}$$

where (x, y, z) is the coordinate on C^3. These surfaces are attractive because every smooth complex projective surface can be obtained as the normalization of such a surface S in the 3-dimensional complex projective space $P^3(C)$. In fact, every smooth, compact complex surface embedded in a complex projective space can be projected onto such a surface S in $P^3(C)$ via generic projection. We denote by D_S

the singular locus of S, and call it the *double curve* of S. D_S is a singular curve with triple points. We denote by Σt_S the triple point locus of S, and by Σc_S the cuspidal point locus of S. Let $f : X \to S$ be the normalization. Note that X is non-singular. We put $D_X := f^{-1}(D_S)$ and $\Sigma t_X := f^{-1}(\Sigma t_S)$. D_X is a singular curve with nodes and Σt_X coincides with the set of nodes of D_X. Let $n_S : D_S^* \to D_S$ and $n_X : D_X^* \to D_X$ be the normalizations, and let $g : D_X^* \to D_S^*$ be the lifting of the map $f_{|D_X} : D_X \to D_S$. We put $\Sigma t_S^* := n_S^{-1}(\Sigma t_S)$ and $\Sigma t_X^* := n_X^{-1}(\Sigma t_X)$. Then a *2-cubic hyper-resolution* of S in the sense of F. Guillén, V. Navarro Aznar *et al.* ([2]) is obtained as in the diagram (1.1) below. In the diagram, ν_S and ν_X are the composites of the normalizations and the inclusion maps, and the square on the left-hand side is the one induced from the square on the right-hand side.

(1.1)



2 Description of $H^1(S, \Theta_S)$ by use of the 2-cubic hyper-resolution of S

We put $\Theta_S := Hom_{\mathcal{O}_S}(\Omega_S^1, \mathcal{O}_S)$, and call it the *sheaf of germs of holomorphic tangent vector fields on S*. We call $H^1(S, \Theta_S)$ the *infinitesimal locally trivial deformation space* of a compact complex surface S with ordinary singularities. This naming is due to the fact that the parameter space of the 1st-order infinitesimal *locally trivial deformation* of S sits in this space, where "*locally trivial deformation*" means the deformation which preserves local analytic singularity types. In the following we shall describe $H^1(S, \Theta_S)$ by use of the diagram (1.1). We denote symbolically the 2-cubic hyper-resolution of S in the diagram (1.1) by $b. : X. \to S$. For each $\alpha \in \mathrm{Ob}(\square_2^+) := \{\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathrm{Z}^3 \mid 0 \le \alpha_i \le 1 \text{ for } 0 \le i \le 2\}$, an object of the *augmented 2-cubic category* in the sense of F. Guillén, V. Navarro Aznar *et al.* ([2]), we denote by Θ_{X_α} the sheaf of germs of holomorphic tangent vector fields on X_α ($X_0 := S$ for $0 := (0,0,0) \in \mathrm{Ob}(\square_2^+)$), and by $\Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ the sheaf of germs of $\mathcal{O}_{X_\alpha}$-valued derivations on S, i.e., $\theta \in \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ is a C-linear map $\mathcal{O}_S \to b_{\alpha *}\mathcal{O}_{X_\alpha}$ with the property $\theta(uv) = \theta(u)v + u\theta(v)$ for $u, v \in \mathcal{O}_S$, where b_α is the map from X_α to S in the diagram (1.1) (cf. [2]). For each $\alpha \in \mathrm{Ob}(\square_2) := \{\alpha \in \mathrm{Ob}(\square_2^+) \mid \alpha \neq (0,0,0)\}$, we define $tb_\alpha : b_{\alpha *}\Theta_{X_\alpha} \to \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$ (resp. $\omega b_\alpha : \Theta_S \to \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha})$) by $tb_\alpha(\theta) := \theta b_\alpha^*$ for $\theta \in b_{\alpha *}\Theta_{X_\alpha}$ (resp. $\omega b_\alpha(\varphi) := b_\alpha^* \varphi$ for $\varphi \in \Theta_S$), where $b_\alpha^* : \mathcal{O}_S \to b_{\alpha *}\mathcal{O}_{X_\alpha}$ denotes the pull-back.

Definition 1 We define

$$\Theta(b.) := Ker\{\oplus_{\alpha \in \mathrm{Ob}(\square_2^+)} b_{\alpha *}\Theta_{X_\alpha} \to \oplus_{\alpha \in \mathrm{Ob}(\square_2)} \Theta(\mathcal{O}_S, \mathcal{O}_{X_\alpha}) : (\theta_\alpha) \to tb_\alpha(\theta_\alpha) - \omega b_\alpha(\theta_0)\},$$

and call it the *sheaf of germs of holomorphic tangent vector fields to the 2-cubic hyper-resolution* $b. : X. \to S$.

Further, we introduce the following notation:

$\Theta_X(-logD_X)$: the sheaf of germs of logarithmic tangent vector fields along D_X on X, i.e., the subsheaf of Θ_X consisting of derivations of $\mathcal{O}_X$ which send $\mathcal{I}(D_X)$, the ideal sheaf of D_X in $\mathcal{O}_X$, into itself.

$\Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)$: the sheaf of germs of holomorphic tangent vector fields on D_S^* which vanish on Σc_S^* and Σt_S^*, where Σc_S^* is the inverse image of the cuspidal point locus Σc_S of S by the normalization map $n_S : D_S^* \to D_S$,

$\Theta_{D_X^*}(-\Sigma t_X^*)$: the sheaf of germs of holomorphic tangent vector fields on D_X^* which vanish on Σt_X^*. (Note that Σt_X^* coincides with the inverse image of the triple point locus Σt_S of D_S by the composed map $n_S \circ g : D_X^* \to D_S$.)

Proposition 1 *There exists naturally the following exact sequence of $\mathcal{O}_S$-modules:*

$$0 \longrightarrow \Theta_S \xrightarrow{\widehat{\omega f} \oplus \widehat{\omega \nu_S}} f_*\Theta_X(-logD_X) \oplus \nu_{S*}\Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)$$

$$\xrightarrow{\widehat{\omega \nu_X} - \widehat{\omega g}} \nu_* \Theta_{D_X^*}(-\Sigma t_X^*)\, 0 \longrightarrow$$

where $\nu := f \circ \nu_X = \nu_S \circ g$.

The proof of this proposition is a direct calculation by use of the local coordinate description of the maps $f : X \to S$, $\nu_S : D_S^* \to S$, $\nu_X : D_X^* \to X$, and $g : D_X^* \to D_S^*$.

Corollary 1 $\Theta(b.) \simeq \Theta_S$.

Theorem 1 *If the map*

$$H^0(X, \Theta_X(-logD_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \to H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is surjective, then we have

$$\begin{aligned} &H^1(S, \Theta(b.)) \simeq H^1(S, \Theta_S) \\ &\simeq Ker\{H^1(X, \Theta_X(-logD_X)) \oplus H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \\ &\qquad\qquad \to H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))\}. \end{aligned}$$

Proposition 2 *The map*

$$H^1(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \to H^1(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is injective.

The proof of this proposition will be completed after a few lemmas. First, we will prove general facts about a double covering $\pi : C_1 \to C$ between compact Riemann surfaces, or connected, compact complex manifolds of dimension 1. We denote by Σc the branch locus of the double covering $\pi : C_1 \to C$, and by $[\Sigma c]$ the line bundle over C determined by the divisor Σc. Due to Wavrik's result ([7]), there exists a complex line bundle F over C such that;

(i) $F^{\otimes 2} = [\Sigma c]$, and

(ii) C_1 is a submanifold of F and the bundle map $F \to C$ realizes the double covering $\pi : C_1 \to C$.

The transition functions of the line bundle F are given as follows: We choose a covering $\{U_j, U_\lambda\}$ of C by polycylinders having the following properties;

(i) $U_j \cap \Sigma c = \emptyset$, and $U_\lambda \cap \Sigma c \neq \emptyset$,

(ii) on U_λ, Σc has the equation $u_\lambda = 0$ where u_λ is a local coordinate on U_λ,

(iii) $\pi^{-1}(U_j) = U_j^{(0)} \cup U_j^{(1)}$, $U_j^{(0)} \cup U_j^{(1)} = \emptyset$,

(iv) on $U_j^{(\nu)}$, $\nu = 0, 1$, the map π is given by $u_j = v_j^{(\nu)}$ where u_j and $v_j^{(\nu)}$ are local coordinates on U_j and $U_j^{(\nu)}$, respectively, and

(v) on $U_\lambda^\sharp := \pi^{-1}(U_\lambda)$, the map π is given by $u_\lambda = v_\lambda^2$ where v_λ is a local coordinate on $U_\lambda^\sharp$.

We define

$$f_{ij} := \begin{cases} 1 & \text{if } U_i^{(0)} \cap U_j^{(0)} \neq \emptyset \\ -1 & \text{if } U_i^{(0)} \cap U_j^{(1)} \neq \emptyset, \end{cases}$$

$$f_{\lambda j} := g_{\lambda j}^{(0)},$$

where $g_{\lambda j}^{(0)}$ denotes the coordinate-transformation function, i.e., $v_\lambda = g_{\lambda j}^{(0)}(v_j^{(0)})$. Then $\{f_{ij}, f_{\lambda j}\}$ are the transition functions of the line bundle F over C. We may think that C_1 is a submanifold of F defined by $\xi_i^2 = 1$ on $\pi^{-1}(U_i)$, and by $\xi_\lambda^2 = u_\lambda$ on $\pi^{-1}(U_\lambda)$ where ξ_i and ξ_λ are fiber coordinates of F over U_i and U_λ, respectively.

Lemma 1 *With the notation above, there exists an exact sequence of $\mathcal{O}_C$-modules*

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{C_1} \to \mathcal{O}_C(F^{-1}) \to 0. \tag{2.1}$$

Proof. We use the same notation as before. The homomorphism $\pi_*\mathcal{O}_{C_1} \to \mathcal{O}_C(F^{-1})$ of $\mathcal{O}_C$-modules is defined as follows: For a local cross-section $(U_i^{(0)}, \phi_i^{(0)})$, $(U_i^{(1)}, \phi_i^{(1)})$ of $\pi_*\mathcal{O}_{C_1}$ over U_i, we put

$$\psi_i(u_i) := \phi_i^{(0)}(u_i) - \phi_i^{(1)}(u_i).$$

For a local cross-section $(U_\lambda^\sharp, \phi_\lambda)$ of $\pi_*\mathcal{O}_{C_1}$ over U_λ, we put

$$\psi_\lambda(u_\lambda) := \frac{\phi_\lambda(v_\lambda) - \phi_\lambda(-v_\lambda)}{v_\lambda}.$$

We note that the right-hand-side of this is invariant by the transformation $v_\lambda \to -v_\lambda$, and so it defines a holomorphic function on U_λ. We can see that the collection

$\{\psi_i, \psi_\lambda\}$ defines a local cross-section of $\mathcal{O}_C(F^{-1})$. Indeed, if $U_i^{(0)} \cap U_j^{(0)} \neq \emptyset$, we have $\phi_i^{(0)} = \phi_j^{(0)}$ on $U_i^{(0)} \cap U_j^{(0)}$, $\phi_i^{(1)} = \phi_j^{(1)}$ on $U_i^{(1)} \cap U_j^{(1)}$ and $f_{ij} = 1$. Hence

$$\psi_i = \phi_i^{(0)} - \phi_i^{(1)} = \phi_j^{(0)} - \phi_j^{(1)} = f_{ij}^{-1}\psi_j \text{ on } U_i \cap U_j.$$

If $U_i^{(0)} \cap U_j^{(1)} \neq \emptyset$, we have $\phi_i^{(0)} = \phi_j^{(1)}$ on $U_i^{(0)} \cap U_j^{(1)}$, $\phi_i^{(1)} = \phi_j^{(0)}$ on $U_i^{(1)} \cap U_j^{(0)}$ and $f_{ij} = -1$. Hence

$$\psi_i = \phi_i^{(0)} - \phi_i^{(1)} = -(\phi_j^{(0)} - \phi_j^{(1)}) = f_{ij}^{-1}\psi_j \text{ on } U_i \cap U_j.$$

If $U_\lambda \cap U_i \neq \emptyset$, we have

$$\begin{cases} \phi_\lambda(v_\lambda) = \phi_i^{(0)}(v_i^{(0)}) & \text{on } U_\lambda^\sharp \cap U_i^{(0)}, \textit{ and} \\ \phi_\lambda(v_\lambda) = \phi_i^{(1)}(v_i^{(1)}) & \text{on } U_\lambda^\sharp \cap U_i^{(1)}. \end{cases}$$

Hence

$$\phi_\lambda(-v_\lambda) = \phi_i^{(1)}(v_i^{(0)}) \text{ on } U_\lambda^\sharp \cap U_i^{(0)}$$

and

$$\psi_\lambda(u_\lambda) = \frac{\phi_\lambda(v_\lambda) - \phi_\lambda(-v_\lambda)}{v_\lambda} = g_{\lambda i}^{(0)}(v_i^{(0)})^{-1}\{\phi_i^{(0)}(v_i^{(0)}) - \phi_i^{(1)}(v_i^{(0)})\}$$
$$= f_{\lambda i}^{-1}(u_i)\psi_i(u_i) \text{ on } U_\lambda \cap U_i.$$

Thus the collection $\{\psi_i, \psi_\lambda\}$ certainly defines a local cross-section of $\mathcal{O}_C(F^{-1})$. We define the homomorphism $\pi_*\mathcal{O}_{C_1} \to \mathcal{O}_C(F^{-1})$ in (2.1) by the correspondence

$$\begin{cases} (\phi_i^{(0)}, \phi_i^{(1)}) \longmapsto \psi_i & \text{over } U_i, \textit{ and} \\ \phi_\lambda \longmapsto \psi_\lambda & \text{over } U_\lambda. \end{cases}$$

The fact that the kernel of the homomorphism $\pi_*\mathcal{O}_{C_1} \to \mathcal{O}_C(F^{-1})$ is $\mathcal{O}_C$ is obvious. The surjectivity of the homomorphism $\pi_*\mathcal{O}_{C_1} \to \mathcal{O}_C(F^{-1})$ at a point $p \notin \Sigma c_\lambda$ is also obvious. We will show the surjectivity of this homomorphism at a point $p \in \Sigma c_\lambda$.

Let ψ be a local cross-section of $\mathcal{O}_C(F^{-1})$ at the point p. We may think of it as a holomorphic function defined around p. Let

$$\psi(u_\lambda) = \Sigma_{k=0}^{\infty} a_k u_\lambda^k$$

be the power series expansion of ψ with center p. We put

$$\phi(v_\lambda) = \Sigma_{k=0}^{\infty} \frac{1}{2} a_k v_\lambda^{2k+1}$$

Then, since $u_\lambda = v_\lambda^2$, we have

$$\psi_\lambda(u_\lambda) = \frac{\phi_\lambda(v_\lambda) - \phi_\lambda(-v_\lambda)}{v_\lambda}.$$

Thus the homomorphism $\pi_*\mathcal{O}_{C_1} \to \mathcal{O}_C(F^{-1})$ is surjective at the point $p \in \Sigma\, c$.

Q.E.D.

Let $\pi : C_1 \to C$ and $\Sigma\, c$ be the same as before, and let Σt be a set of finite distinct points of C with $\Sigma\, c \cap \Sigma t = \emptyset$. We put $\Sigma t_1 := \pi^{-1}(\Sigma t)$.

Lemma 2 *With the notation above, we have an exact sequence of $\mathcal{O}_C$-modules*

(2.2) $$0 \to \Theta_C(-\Sigma\, c - \Sigma t) \to \pi_*\Theta_{C_1}(-\Sigma t_1) \to \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}) \to 0\,.$$

Proof Since $\pi_*(\pi^*\Theta_C(-\Sigma t)) \simeq \Theta_C(-\Sigma t) \otimes \pi_*\mathcal{O}_{C_1}$, tensoring the sheaf $\Theta_C(-\Sigma t)$ to the exact sequence in (2.1), we have an exact sequence of $\mathcal{O}_C$-modules

(2.3) $$0 \to \Theta_C(-\Sigma t) \to \pi_*(\pi^*\Theta_C(-\Sigma t)) \to \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1}) \to 0.$$

We also have the following commutative diagram of exact sequences of $\mathcal{O}_C$-modules:

$$\begin{array}{ccccc}
0 & \to & \Theta_C(-\Sigma t) & \xrightarrow{\widehat{\omega\pi}} & \pi_*(\pi^*\Theta_C(-\Sigma t)) \\
 & & \uparrow & & \uparrow \\
0 & \to & \Theta_C(-\Sigma\, c - \Sigma t) & \to & \pi_*\Theta_{C_1}(-\Sigma t_1) \\
 & & \uparrow & & \uparrow \\
 & & 0 & & 0
\end{array}$$

where $\widehat{\omega\pi}$ denotes the pull-back. We will show that this diagram gives an isomorphism

$$(2.4)\qquad \pi_*\Theta_{C_1}(-\Sigma t_1)/\Theta_C(-\Sigma c-\Sigma t)\simeq\pi_*(\pi^*\Theta_C(-\Sigma t))/\Theta_C(-\Sigma t).$$

To prove the surjectivity of the homomorphism in (2.4), we will first show that

$$(2.5)\qquad \widehat{t\pi}(\Theta_{C_1}(-\Sigma t_1)_{\pi^{-1}(p)})+\widehat{\omega\pi}(\Theta_C(-\Sigma t)_p)=\pi^*\Theta_C(-\Sigma t)_{\pi^{-1}(p)}$$

for any point $p\in C$, where $\widehat{t\pi}$ denotes the map derived from the Jacobian map of the map π. If $p\notin\Sigma c$, (2.5) obviously holds. Assume $p\in\Sigma c$. We put $q:=\pi^{-1}(p)$, and let u and v be local coordinates around p and q with center p and q, respectively. We may assume that the map $\pi: C_1\to C$ is given by $v\to u=v^2$ at q. For a local cross-section $a(v)\pi^*(\partial/\partial u)$ of $\pi^*\Theta_C(-\Sigma t)$ around q where $a(v)$ is a holomorphic function of v, we express $a(v)$ as

$$a(v)=a(0)+va_1(v)$$

where $a_1(v)$ is a holomorphic function of v. Then we have

$$\begin{aligned}&\widehat{t\pi}(\frac{1}{2}a_1(v)(\frac{\partial}{\partial v}))+\widehat{\omega\pi}(a(0)(\frac{\partial}{\partial u}))\\&=(va_1(v)+a(0))\pi^*(\frac{\partial}{\partial u})=a(v)\pi^*(\frac{\partial}{\partial u}),\end{aligned}$$

which shows that (2.5) holds for the point $p\in\Sigma c$. To prove the injectivity of the homomorphism in (2.4), it suffices to show that, for any point $p\in C$ and a local holomorphic cross-section θ_1 of $\pi_*\Theta_{C_1}(-\Sigma t_1)$ at p, if $\widehat{t\pi}(\theta_{1,p})$ belongs to $\widehat{\omega\pi}(\Theta_C(-\Sigma t)_p)$, then $\theta_{1,p}$ belongs to the image $\Theta_C(-\Sigma c-\Sigma t)_p$ in $\pi_*\Theta_{C_1}(-\Sigma t_1)_p$. Since this is obvious if $p\notin\Sigma c$, we assume $p\in\Sigma c$. We take the same local coordinates u and v around p and $q:=\pi^{-1}(p)$ as before, respectively. For a local cross-section $\theta_1=a_1(v)(\partial/\partial v)$ of $\Theta_{C_1}(-\Sigma t_1)$ at q, we assume that there exists a local cross-section $\theta=a(u)(\partial/\partial u)$ of $\Theta_C(-\Sigma t)$ at p such that $\widehat{t\pi}(\theta_1)=\widehat{\omega\pi}(\theta)$. Then

$$2a_1(v)v\pi^*(\frac{\partial}{\partial v})=a(v^2)\pi^*(\frac{\partial}{\partial v})$$

Hence $a(0)=0$, that is, θ belongs to $\Theta_C(-\Sigma c-\Sigma t)$. This means θ_1 belongs to the image of $\Theta_C(-\Sigma c-\Sigma t)$ in $\pi_*\Theta_{C_1}(-\Sigma t_1)$ at p. Now the exact sequence in (2.2) follows from (2.3) and (2.4).

Q.E.D.

Remark 1 In the proof of Lemma 2, the equality in (2.5) is essential. This equality tells that the double branched covering map $\pi : C_1 \to C$ is *locally stable* in the sense of J. N. Mather (cf. [1]).

Proof of Proposition 2 We may assume that D_S^* is irreducible, and so it suffices to show that the homomorphism

$$H^1(C, \Theta_C(-\Sigma c - \Sigma t)) \to H^1(C_1, \Theta_{C_1}(-\Sigma t_1)) \tag{2.6}$$

derived from the exact sequence in (2.2) is injective. For this purpose, we count the degree of the line bundle $\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})$. We denote by K_C and $g(C)$ the canonical line bundle and the genus of the curve C, respectively. Then, since $F^{\otimes 2} = \mathcal{O}_C([\Sigma c])$, we have

$$\begin{aligned}\deg(\Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) &= -\deg K_C - \deg F - \sharp \Sigma t \\ &= -2(g(C) - 1) - \frac{1}{2}\sharp \Sigma c - \sharp \Sigma t,\end{aligned}$$

where $\sharp$ denote the cardinal numbers of sets. Then we have

$$-2(g(C) - 1) - \frac{1}{2}\sharp \Sigma c - \sharp \Sigma t < 0$$

with the exception of the following cases:

(i) $g(C) = 1$, $\Sigma c = \emptyset$, and $\Sigma t = \emptyset$,

(ii) $g(C) = 0$, $\Sigma c = \emptyset$, and $0 \leq \sharp \Sigma t \leq 2$,

(iii) $g(C) = 0$, $\sharp \Sigma c = 2$, and $0 \leq \sharp \Sigma t \leq 1$,

(iv) $g(C) = 0$, $\sharp \Sigma c = 4$, and $\Sigma t = \emptyset$.

Hence, excluding the exceptional cases listed above, we have

$$H^0(C, \Theta_C(-\Sigma t) \otimes \mathcal{O}_C(F^{-1})) = 0, \tag{2.7}$$

and so the homomorphism in (2.6) is injective as required. Now, checking the exceptional cases, case by case, we conclude that the homomorphism in (2.6) is always injective.

Corollary 2 *If the map*

(2.8)
$$H^0(X, \Theta_X(-logD_X)) \oplus H^0(D_S^*, \Theta_{D_S^*}(-\Sigma c_S^* - \Sigma t_S^*)) \to H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*))$$

is surjective, then the natural map

$$H^1(S, \Theta_S) \to H^1(X, \Theta_X(-logD_X))$$

is injective.

By Cororally 2, we have the following:

Theorem 2 *For a compact complex surface S with ordinary singularities, we denote by M (resp. M_1) the parameter space of the Kuranishi family of locally trivial deformations of S (resp. of the pair (X, D_X)), and by o (resp. o_1) the point of M (resp. M_1) corresponding to the surface S (resp. the pair (X, D_X)). If the map in (2.8) is surjective, then there exists a closed embedding from a sufficiently small open neighborhood of o in M into that of o_1 in M_1 with $h(o) = o_1$.*

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