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LOCALLY STABLE HOLOMORPHIC MAPS AND THEIR APPLICATION TO A GLOBAL MODULI PROBLEM FOR SOME KINDS OF ANALYTIC SUBVARIETIES

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Introduction

Classically, an important way of studying a complex projective algebraic manifold X^n of dimension n is to map X^n birationally onto a hypersurface in $P^{n+1}(\mathbb{C})$. Such a map $f\colon X^n \to P^{n+1}(\mathbb{C})$ can be obtained via a linear projection from the center in general position. In the cases where $n = 2, 3$, the singularities of $f(X^n)$ are described as follows:

In the case where $n = 2$ [20]:

$$\begin{cases} \text{(i)} & yz = 0, \\ \text{(ii)} & xyz = 0, \\ \text{(iii)} & xy^2 - z^2 = 0. \end{cases}$$

In the case where $n = 3$ [15]:

$$\begin{cases} \text{(i)} & zw = 0, \\ \text{(ii)} & yzw = 0, \\ \text{(iii)} & xyzw = 0, \\ \text{(iv)} & xy^2 - z^2 = 0, \\ \text{(v)} & w(xy^2 - z^2) = 0. \end{cases}$$

Here (x, y, z) and (x, y, z, w) denote adequate analytic local coordinate systems around singular points. The singularities described above are known as *ordinary singularities*. The singularity $xy^2 - z^2 = 0$ is known as an *ordinary cuspidal point* or a *Whitney umbrella*.

In 1961, Kodaira proved the following [9].

Let S be an analytic surface with ordinary singularities in a nonsingular, compact complex threefold W. If S is *semiregular*, then there exists an *effectively parametrized, maximal* (=semi-universal) family $\mathfrak{S} = \{S_t\}_{t\in M}$ of *locally trivial* displacements of S in W, parametrized by a complex manifold germ $(M, 0)$.

In 1973, Mather showed that, as concerns generic linear projections, the complex analytic analogue of his theory on C^∞-stable maps is possible [14]. As a result, it turned out that the description of ordinary singularities of surfaces and threefolds given above is merely a very small part of a deeper and wider theory. By [13], [10, Theorem 5.1], and [11, Theorem 4.1], we can say that if we define an *n-dimensional ordinary singularity in an m-dimensional ambient manifold* as a germ of an analytic subvariety (Y, Z, q) $(q \in Z \subset Y)$, for which there exists a simultaneously stable multigerm $f\colon (X, S) \to (Y, q)$ of a holomorphic map from an n-dimensional complex manifold X into an m-dimensional one Y with the property $Z = f(X)$, then there exists a routine of giving the defining equation of such a singularitity, at least when the pair of integers (n, m) belongs to the so-called "nice range" (cf. [12]) and satisfies $n < m$. Here we note that, in the case $m = n + 1$, this occurs if and only if $n \leq 14$. Following this routine, we showed in [16] that, besides the cases of dimension 2 and 3, the defining equations of hypersurface ordinary singularities of dimension 4 and 5 could be actually calculated.

Inspired by Mather's works on stable maps, we introduced in [17] the notion of *analytic subvarieties with locally stable parametrizations* of complex manifolds (cf. Definition 2.1 below), which is a unification and a generalization of closed complex subsets of normal crossing (not necessarily of pure dimension) and analytic

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subvarieties with ordinary singularities, and showed that we could *globalize* Kodaira's result stated above and generalize it to higher-dimensional cases without the semiregularity condition. The purpose of this article is to give an outline of our result. We can also announce that there naturally arise variations of a mixed Hodge structure from the locally trivial families of surfaces and threefolds with ordinary singularities at smooth points of the parameter spaces.

Throughout the article, we use the terms *analytic variety* and *analytic subvariety* in the sense of a reduced complex space and a reduced closed complex subspace, respectively; and by families of deformations of holomorphic maps or by families of displacements of analytic subvarieties we always mean families parametrized by possibly nonreduced complex spaces, unless otherwise specified.

1. Locally Stable Holomorphic Maps

We shall review the basic facts concerning locally stable holomorphic maps.

Definition 1.1. The multigerm $f : (X,S) \to (Y,q)$ of a holomorphic map with $f(S) = q$ is said to be *simultaneously stable* if any small deformation of f is trivial.

Definition 1.2. The holomorphic map $f: X \to Y$ between complex manifolds is said to be *locally stable* if, for any point $q \in f(X)$ and any finite subset $S = \{p_1, \cdots, p_s\} \subset f^{-1}(q)$, the multigerm $f: (X,S) \to (Y,q)$ is simultaneously stable.

We denote the sheaf of germs of holomorphic vector fields on X (resp. Y) by Θ_X (resp. Θ_Y), and the pull-back of Θ_Y by a holomorphic map $f: X \to Y$ by $f^*\Theta_Y$. We define $\Theta_{X,S} := \Theta_{X,p_1} \times \ldots \times \Theta_{X,p_s}$ and, similarly, $f^*\Theta_{Y,S}$, $\mathcal{O}_{X,S}$. We denote by $tf: \Theta_{X,S} \to f^*\Theta_{Y,S}$ the $\mathcal{O}_{X,S}$-homomorphism defined by the Jacobian map of f, and by $\omega f: \Theta_{Y,q} \to f^*\Theta_{Y,S}$ the homomorphism over f^* defined by the pull-back by f, where f^* denotes the homomorphism $\mathcal{O}_{Y,q} \to \mathcal{O}_{X,S}$ induced by f.

Theorem 1.3 ([16, Theorem A in Appendix, Remark 2.1] and [13]). *The multigerm $f: (X,S) \to (Y,q)$ of a holomorphic map is simultaneously stable if and only if*

$$tf(\Theta_{X,S}) + \omega f(\Theta_{Y,q}) = f^*\Theta_{Y,S}.$$

Let U_i $(1 \le i \le s)$ be a connected open subset in $\mathbb{C}^{n_i}$. We put $U := \coprod_{i=1}^s U_i$, the disjoint union of $U_1, \ldots, U_s$. Let $S = \{p_1, \ldots, p_s\}$ be a finite subset of $\coprod_{i=1}^s \mathbb{C}^{n_i}$ with $p_i \in U_i$ for $1 \le i \le s$. We denote by $\Delta(n_i, p_i, r)$ the polydisk in $\mathbb{C}^{n_i}$ with center p_i and radius r, and by $\mathrm{Hol}\,(U, \mathbb{C}^m)$ the set of all holomorphic maps from U into $\mathbb{C}^m$.

Theorem 1.4 ([16, Theorem B in Appendix]). *Let $f: U \to \mathbb{C}^m$ be a holomorphic map with $f(S) = O$, the origin of $\mathbb{C}^m$. If the multigerm $f: (U,S) \to (\mathbb{C}^m, O)$ is simultaneously stable, then, for any relative compact neighborhood U' of S in U of the form $U' = \coprod_{i=1}^s \Delta(n_i, p_i, r)$, there exists an open neighborhood*

$$N_\varepsilon(U') = \{g \in \mathrm{Hol}\,(U, \mathbb{C}^m) |\ \sup_{x \in U'} |f(x) - g(x)| < \varepsilon\}$$

of f in $\mathrm{Hol}\,(U, \mathbb{C}^m)$, where ε is a suitably chosen positive number, with the following property.

For any $g \in N_\varepsilon(U')$ there exists a quadruple (U'', W, ϕ, ψ), where $U'' := \coprod_{i=1}^s U_i''$ is the disjoint union of connected open subsets $U_i'' \subset \Delta(n_i, p_i, r)$ with $p_i \in U_i''$ $(1 \le i \le s)$, W is an open neighborhood of the origin O in $\mathbb{C}^m$, ϕ is an analytic embedding (i.e., invertible holomorphic map) $U'' \to U'$ of the form $\phi := \coprod_{i=1}^s \phi_i$, the disjoint union of analytic embeddings $\phi_i: U_i'' \to \Delta(n_i; p_i; r)$ $(1 \le i \le s)$, and ψ is an analytic embedding $W \to \mathbb{C}^m$ such that the following diagram commutes:

$$\begin{array}{ccc} U'' & \xrightarrow{\phi} & U \\ f|_{U''} \downarrow & & g \downarrow \\ W & \xrightarrow{\psi} & \mathbb{C}^m . \end{array}$$

Corollary 1.5 (**[16, Theorem 4.2]**). *A locally stable holomorphic map is a Thom-Boardman map satisfying a normal crossing condition* (see [6, Chapter VI, Sec. 5] for the definition).

Corollary 1.6 (**[16, Proposition 4.2]**). *If $f\colon X \to Y$ is a proper locally stable holomorphic map between complex manifolds with $\dim X < \dim Y$ and if $X \xrightarrow{f'} Z \hookrightarrow Y$ is the factorization of f, where $Z := f(X)$, then $f'\colon X \to Z$ is a normalization map of Z.*

Theorem 1.7 (**[18, Theorem 2.4]**). *If $f\colon X \to Y$ is a locally stable holomorphic map between compact complex manifolds, then any small deformation of f is locally stable.*

Theorem 1.8 (**[18, Theorem 3.4]**). *Let $\mathfrak{F} := (\mathfrak{X}, F, \mathfrak{Y}, \pi_1, \pi_2, D, O, \phi, \psi)$ be a complex analytic family of deformations of the locally stable holomorphic map $f : X \to Y$ between compact complex manifolds, parametrized by the domain D of $\mathbb{C}^m$ with the origin $0 \in D$. Then the ambient C^∞ family $\mathfrak{F}_{\mathbb{R}}$ of C^∞ maps of the family $\mathfrak{F}$ is C^∞ trivial at the origin $0 \in D$.*

2. Analytic Subvarieties with Locally Stable Parametrizations of a Compact Complex Manifold and Their Global Moduli

Definition 2.1. The analytic subvariety Z (possibly not of pure dimension) of the complex manifold Y is said to be *with a locally stable parametrization* (we write "with l.s.p." for short) if

(i) its normal model X is nonsingular, and

(ii) the composite map $f := \iota \circ n\colon X \to Y$ is locally stable, where $n\colon X \to Z$ is a normalization map and $\iota\colon Z \hookrightarrow Y$ is an inclusion map.

With this definition we can state our main theorem as follows.

Theorem 2.2 (**[17, Theorem 4.8]**). *Let Y be a compact complex manifold. We denote the set of all analytic subvarieties with l.s.p. of Y by $E(Y)$ and the analytic subvariety with l.s.p. of Y corresponding to the "point" $t \in E(Y)$ by Z_t. We define the subset $\mathfrak{Z}(Y)$ of the product space $Y \times E(Y)$ by*

$$\mathfrak{Z}(Y) := \{(y,t) |\, t \in E(Y),\ y \in Z_t\}.$$

We denote by $\Pi\colon \mathfrak{Z}(Y) \to E(Y)$ the restriction of the projection map $\operatorname{Pr}_{E(Y)}\colon Y \times E(Y) \to E(Y)$ to $\mathfrak{Z}(Y)$. Then $E(Y)$ and $\mathfrak{Z}(Y)$ have the structure of Hausdorff complex spaces such that the family $\Pi\colon \mathfrak{Z}(Y) \to E(Y)$ is universal for locally trivial families of analytic subvarieties with l.s.p. of Y. (Here a locally trivial family, say, $\pi : \mathfrak{Z} \to M$, is locally the product of a fiber and the basic space at every point of the total space $\mathfrak{Z}$). Furthermore, the underlying C^∞ structure of the family $\Pi\colon \mathfrak{Z}(Y) \to E(Y)$ is C^∞ trivial at a nonsingular point of $E(Y)_{\text{red}}$ (the reduction of $E(Y)$).

Outline of the proof. We shall begin by proving that we can simultaneously normalize the given locally trivial family $\pi\colon \mathfrak{Z} \to M$ of analytic subvarieties with l.s.p. of Y, parametrized by the complex space M (see [17, Theorem 3.6]). It follows that the category of locally trivial families of analytic subvarieties with l.s.p. of Y is equivalent to that of families of locally stable holomorphic maps of compact complex manifolds, whose dimensions are smaller than that of Y, into Y. Then the theorem can be proved with the use of the following three results: the result of Flenner concerning the existence of a semi-universal family for deformations of holomorphic maps between compact complex spaces [4], the result of Flenner and Kosarew concerning the existence of the maximal family for locally trivial deformations of complex space germs [5], and the result of Douady concerning the existence of global moduli of compact closed complex subspaces of a complex manifold [2]. The C^∞ triviality of the family $\Pi\colon \mathfrak{Z}(Y) \to E(Y)$ follows from Theorem 1.8 in Sec. 1. For details, see [17].

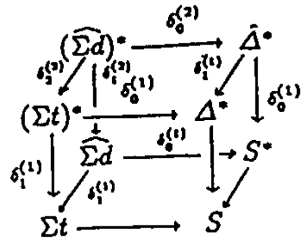

Fig. 1

3. Variations of a Mixed Hodge Structure Arising from the Locally Trivial Families of Algebraic Surfaces and Threefolds with Ordinary Singularities

Since we have only a little space left, we shall explain our result only for the case of surfaces. The proof for the case of threefolds is similar. Let S be a hypersurface with ordinary singularities in a nonsingular complex projective algebraic threefold. The cubic hyperresolution of S in the sense of Guillén, Navarro Aznar, Pascual-Gainza, and Puerta is as shown in Fig. 1 (cf. [8]). Here S^* is a normal model of S, Δ^* is a normal model of the double curve Δ of S, Σt is the triple-points locus of S, $\hat{\Delta}^*$ the normal model of the inverse image $\hat{\Delta}$ of the double curve Δ by the normalization map $S^* \to S$, $(\Sigma t)^*$ is the inverse image of Σt by the normalization map $\Delta^* \to \Delta$, and $(\widehat{\Sigma d})^*$ is the inverse image of the double-points locus $\widehat{\Sigma d}$ of $\hat{\Delta}$ by the normalization map $\hat{\Delta}^* \to \hat{\Delta}$. The semisimplicial hyperresolution of S associated with this cubic hyperresolution is as follows (cf. [7]):

$$X_2 \begin{array}{c} \xrightarrow{\delta_0^{(2)}} \\ \xrightarrow{\delta_1^{(2)}} \\ \xrightarrow[\delta_2^{(2)}]{} \end{array} X_1 \begin{array}{c} \xrightarrow{\delta_0^{(1)}} \\ \xrightarrow{\delta_1^{(1)}} \end{array} X_0 \xrightarrow{\delta^{(0)}} S, \tag{$*$}$$

where $X_0 := \Sigma t \coprod \Delta^* \coprod S^*$ (disjoint union), $X_1 := (\Sigma t)^* \coprod \widehat{\Sigma d} \coprod \hat{\Delta}^*$, and $X_2 := (\widehat{\Sigma d})^*$. We put $\pi_2 := \Delta^{(0)} \circ \Delta_{i_1}^{(1)} \circ \Delta_{i_2}^{(2)} : X_2 \to S$, $\pi_1 := \Delta^{(0)} \circ \Delta_{i_1}^{(1)} : X_1 \to S$, and $\pi_0 := \Delta^{(0)} : X_0 \to S$, where $i_1 = 0, 1$ and $i_2 = 0, 1, 2$. We denote by $D^+(S, \mathbb{Z})$ the derived category of lower bounded complexes of sheaves of $\mathbb{Z}$-modules over S. We define $K \in \mathrm{Ob}\,(D^+(S, \mathbb{Z}))$ by

$$K : 0 \to \pi_{0*}\mathbb{Z}_{X_0} \xrightarrow{d^0} \pi_{1*}\mathbb{Z}_{X_1} \xrightarrow{d^1} \pi_{2*}\mathbb{Z}_{X_2} \to 0,$$

where $d^0 := \Delta_0^{(1)*} - \Delta_1^{(1)*}$ and $d^1 := \Delta_0^{(2)*} - \Delta_1^{(2)*} + \Delta_2^{(2)*}$. Then $K = \mathbb{Z}_S$ in $D^+(S, \mathbb{Z})$. We define the so-called *weight filtration* W on $K_{\mathbb{Q}} := K \otimes \mathbb{Q} \in \mathrm{Ob}\,(D^+(S, \mathbb{Q}))$ by $W_{-q}(K_{\mathbb{Q}}) := \sigma_{\cdot \geq q}\pi_{\cdot *}\mathbb{Q}_{X_\cdot}$ (*stupid filtration*). Then $(K_{\mathbb{Q}}, W) \in \mathrm{Ob}\,(D^+F(S, \mathbb{Q}))$, where $D^+F(S, \mathbb{Q})$ denotes the derived category of filtered, lower bounded complexes of sheaves of $\mathbb{Q}$-modules over S. We can prove by calculation that $K_{\mathbb{C}} := K \otimes \mathbb{C}$ is quasi-isomorphic to $s(\pi_{\cdot *}\omega^{\cdot}_{X_\cdot})$, where $\omega^{\cdot}_{X_i}$ $(i = 0, 1, 2)$ denotes the holomorphic de Rham complex over X_i and $s(\pi_{\cdot *}\omega^{\cdot}_{X_\cdot})$ is a simple complex associated with the double complex $\pi_{\cdot *}\omega^{\cdot}_{X_\cdot}$. We define the so-called *Hodge filtration* F on $K_{\mathbb{C}} \simeq s(\pi_{\cdot *}\omega^{\cdot}_{X_\cdot})$ by $F^p(s(\pi_{\cdot *}\omega^{\cdot}_{X_\cdot})) := s(\sigma_{q \geq p}\pi_{\cdot *}\omega^q_{X_\cdot})$.

Proposition 3.1. *The data* $\mathbb{Z}_S$, $(\pi_{\cdot *}\mathbb{Q}_{X_\cdot}, W)$, $\mathbb{Q}_S \simeq \pi_{\cdot *}\mathbb{Q}_{X_\cdot}$, $(s(\pi_{\cdot *}\omega^{\cdot}_{X_\cdot}), W, F)$, $(\pi_{\cdot *}\mathbb{Q}_{X_\cdot}, W) \otimes \mathbb{C} \simeq (s(\pi_{\cdot *}\omega^{\cdot}_{X_\cdot}), W)$ *is a cohomological mixed Hodge complex in the sense of Deligne* ([1, (8.1.6)]). *Hence it defines a mixed Hodge structure on* $H^n(S, \mathbb{C}) \simeq H^n(\mathbb{R}\Gamma(s(\pi_{\cdot *}\omega^{\cdot}_{X_\cdot})))$ $(0 \leq p \leq \dim S)$.

The corresponding statement in the relative case is as follows.

Theorem 3.2. *Suppose that $f: \mathfrak{X} \to M$ is a locally trivial family of algebraic surfaces (or threefolds) with ordinary singularities in a compact projective algebraic manifold, parametrized by the complex manifold M, and $\pi_\cdot : \mathfrak{X}_\cdot \to \mathfrak{X}$ is a semisimplicial hyperresolution of $\mathfrak{X}$ over M corresponding to $(*)$. We set $R^n_{\mathbb{Z}}(f) := R^n f_* \mathbb{Z}$ modulo torsion, $R^n_{\mathbb{Q}}(f) := R^n_{\mathbb{Z}}(f) \otimes \mathbb{Q}$, and $R^n_{\mathcal{O}}(f) := R^n f_*(f^{\cdot}\mathcal{O}_M) = (R^n f_* \mathbb{Z}) \otimes \mathcal{O}_M$, where $f^{\cdot}\mathcal{O}_M := \mathfrak{X} \times_M \mathcal{O}_M$. Then there exist a weight filtration W on $R^n_{\mathbb{Q}}(f)$ (i.e., a family of increasing sublocal systems) and a Hodge filtration F on $R^n_{\mathcal{O}}(f)$ (i.e., a family of decreasing holomorphic subbundles) meeting the following requirements:*

(MH1) *the Gauss–Mannin connection ∇ on $R^n_{\mathcal{O}}(f)$ satisfies*

$$\nabla F^p(R^n_{\mathcal{O}}(f)) \subset \Omega^1_M \otimes F^{p-1}(R^n_{\mathcal{O}}(f));$$

(MH2) $(R^n_{\mathbb{Z}}(f), W[n], F)(t) \simeq (H^n(X_t, \mathbb{Z}), W[n], F)$ *for any $t \in M$ $(W[n]_q := W_{q-n})$;*

(MH3) *the spectral sequences with respect to the filtrations W and F are described as follows:*

$${}_W E_1^{p,q} = \mathbb{R}^{p+q} f_*(\mathrm{Gr}^W_{-p}(s(\pi_{\cdot *}\omega^{\cdot}_{\mathfrak{X}_\cdot/M}))) \Rightarrow_W E_\infty^{p+q} = \mathbb{R}^{p+q} f_*(s(\pi_{\cdot *}\omega^{\cdot}_{\mathfrak{X}_\cdot/M})),$$

$${}_F E_1^{p,q} = \mathbb{R}^{p+q} f_*(\mathrm{Gr}^p_F(s(\pi_{\cdot *}\omega^{*}_{\mathfrak{X}_\cdot/M}))) \Rightarrow_F E_\infty^{p+q} = \mathbb{R}^{p+q} f_*(s(\pi_{\cdot *}\omega^{\cdot}_{\mathfrak{X}_\cdot/M})),$$

and ${}_W E_2 = {}_W E_\infty$, ${}_F E_1 = {}_F E_\infty$.

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