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The Euler number of the normalization of a certain hypersurface with quasi-ordinary singularities

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Abstract

In [T2] and [T3] we have proved a numerical formula which gives the Euler number of the (non-singular) normalization X of an algebraic threefold with ordinary singularities $\overline{X}$ in $P^4(\mathbf{C})$. In the proof of this formula, we have used a Lefschetz pencil of hyperplane sections on $\overline{X}$, and calculated the *Segre classes* of the singular subscheme of $\overline{X}$ in order to compute the *class* (number) of $\overline{X}$, i.e. the degree of the top *Mather class* of $\overline{X}$ in $P^4(\mathbf{C})$. In this article we will show that this method also works for a wider class of hypersurfaces in $P^4(\mathbf{C})$ to compute the Euler number of their normalizations.

1 An example of a hypersurface with quasi-ordinary singularities in $P^4(\mathbf{C})$

Let H_i $(1\leq i\leq 3)$ be non-singular hypersurfaces of degrees r_i $(1\leq i\leq 3)$, respectively, in the complex projective 4-space $\mathrm{P}^4(\mathbf{C})$ such that they are in general position at every point where they intersect. Let f_i $(1\leq i\leq 3)$ be the homogeneous polynomial of degree r_i which defines the hypersurface H_i. We may assume $r_1 \geq r_2 \geq r_3$ because of symmetry. We choose and fix a positive integer n with $n \geq 2r_1 + 2r_2$. Let $\overline{X}$ be a hypersurface in $\mathrm{P}^4(\mathbf{C})$ defined by the equation

$$F := Af_1f_2f_3 + B(f_1f_2)^2 + C(f_2f_3)^2 + D(f_3f_1)^2 = 0, \qquad (1.1)$$

where A, B, C and D are homogeneous polynomials of five variables of respective degrees $n-r_1-r_2-r_3$, $n-2r_1-2r_2$, $n-2r_2-2r_3$ and $n-2r_3-2r_1$.

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We put $D_{\overline{X}}^{(ij)} := H_i \cap H_j$ $(1 \leq i < j \leq 3)$ and $D_{\overline{X}} := \bigcup_{1 \leq i < j \leq 3} D_{\overline{X}}^{(ij)}$. Then, by Bertini's theorem, $\overline{X}$ is non-singular outside $D_{\overline{X}}$ if we choose sufficiently generic A, B, C and D.

Proposition 1.1. *If the homogeneous polynomials A, B, C and D are chosen sufficiently generic, then $\overline{X}$ is locally isomorphic to one of the following germs of three-dimensional hypersurface singularities at the origin of $\mathbf{C}^4$ at every point of $\overline{X}$:*

(*i*) $w = 0$ (*simple point*),
(*ii*) $zw = 0$ (*ordinary double point*),
(*iii*) $yzw = 0$ (*ordinary triple point*),
(*iv*) $xy^2 - z^2 = 0$ (*cuspidal point*),
(*v*) $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$ (*degenerate ordinary triple point*),

where (x, y, z, w) are the coordinates on $\mathbf{C}^4$.

For the proof we refer to [T1].

2 The singularity $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$

We consider the following affine threefold:

$$\mathbf{C}^4 \supset T : f := (xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0 \tag{2.1}$$

where (x, y, z, w) are the coordinates on $\mathbf{C}^4$. As shown in [T1], T has an ordinary triple point at $(0, 0, 0, w)$ if $w \neq 0$. Hence, we may think of the singularity $(T, 0)$ of T at the origin of $\mathbf{C}^4$ as a *degenerate ordinary triple point*.

Normalization: Let

$$P^3(\mathbf{C}) \supset S : f := (xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0 \tag{2.2}$$

be the hypersurface in $P^3(\mathbf{C})$, defined by the same polynomial f that defines T. This surface S is classically known as the *Steiner surface*. The surface S is obtained by projecting $P^2(\mathbf{C})$ embedded in $P^5(\mathbf{C})$ by the 2-fold Veronese map to $P^3(\mathbf{C})$. Indeed, if we denote by V the image of $\mathrm{P}^2(\mathbf{C})$ in $\mathrm{P}^5(\mathbf{C})$ by the map $v : \mathrm{P}^2(\mathbf{C}) \to \mathrm{P}^5(\mathbf{C})$ defined by

$$\begin{aligned}(\xi_0 : \xi_1 : \xi_2) \in \mathrm{P}^2(\mathbf{C}) \quad \mapsto \quad & (\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_1\xi_2) \\ & = (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathrm{P}^5(\mathbf{C}),\end{aligned}$$

then the surface S coincides with the image of V by the linear projection $\overline{p} : \mathrm{P}^5(\mathbf{C}) \to \mathrm{P}^3(\mathbf{C})$ defined by

$$(x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathrm{P}^5(\mathbf{C}) \mapsto (y_0 : y_1 : y_2 : -(x_0 + x_1 + x_2)) = (x : y : z : w) \in \mathrm{P}^3(\mathbf{C}). \quad (2.3)$$

Applying the calculation similar to that in [T1], we can see that S is an algebraic surface with *ordinary singularities*, whose singular locus D_S is $\{x = y = 0\} \cup \{y = z = 0\} \cup \{z = x = 0\}$, and that S has one ordinary triple point at $[0 : 0 : 0 : 1]$, six cuspidal points at $[0 : 0 : \pm 2 : 1], [0 : \pm 2 : 0 : 1], [\pm 2 : 0 : 0 : 1]$, and ordinary double points at other points of D_S. We denote by C_S the *cone* over S, which is nothing but T. We denote by C_V the cone over V. Since V is a non-singular, *projectively normal* subvariety in $\mathrm{P}^5(\mathbf{C})$, $(C_V, 0)$ is a *normal* singular point (cf. [H], Exercise 3.4 (e), p.394). Hence, if we denote by $p : \mathbf{C}^6 \to \mathbf{C}^4$ the linear projection induced by $\overline{p} : \mathrm{P}^5(\mathbf{C}) \to \mathrm{P}^3(\mathbf{C})$ in (2.3), and by $n : C_V \to C_S (= T)$ the restriction of p to C_V, then $n : C_V \to C_S$ gives the normalization of $(T, 0)$. $(C_V, 0)$ becomes non-singular after a single blowing-up. Indeed, if we denote by τ the blowing-up $\widehat{\mathbf{C}^6} \to \mathbf{C}^6$ at the origin of $\mathbf{C}^6$, $\widehat{\mathbf{C}^6}$ can be identified with $[H_{\mathrm{P}^5(\mathbf{C})}]^{-1}$, where $[H_{\mathrm{P}^5(\mathbf{C})}]$ denotes the line bundle on $\mathrm{P}^5(\mathbf{C})$ determined by a hyperplane $H_{\mathrm{P}^5(\mathbf{C})}$ in $\mathrm{P}^5(\mathbf{C})$. Furthermore, the proper inverse image $\widehat{C_V}$ of C_V by τ (resp. the exceptional divisor $E := \tau^{-1}(0)$) can be identified with $[H_{\mathrm{P}^5(\mathbf{C})}]^{-1}_{|V} \simeq [H_{\mathrm{P}^2(\mathbf{C})}]^{-2}$ (resp. the zero cross-section of the line bundle $L := [H_{\mathrm{P}^2(\mathbf{C})}]^{-2}$ on $\mathrm{P}^2(\mathbf{C})$), where $[H_{\mathrm{P}^5(\mathbf{C})}]^{-1}_{|V}$ denotes the restriction of $[H_{\mathrm{P}^5(\mathbf{C})}]^{-1}$ to V. From this fact, it follows that $E^2 = -2H_{\mathrm{P}^2(\mathbf{C})}$, where $H_{\mathrm{P}^2(\mathbf{C})}$ denotes a hyperplane in $\mathrm{P}^2(\mathbf{C})$.

Theorem 2.1. $(C_V, 0)$ *is*

(i) *rational, and so Cohen-Macaulay,*

(ii) *"rigid" under small deformations,*

(iii) *Gorenstein of index two,*

(iv) *terminal, and so canonical,*

(v) *quasi-ordinary, that is there is a finite morphism $(C_V, 0) \to \mathbf{C}^3$ whose branching locus is contained in the hypersurface of $\mathbf{C}^3$ defined by $x_1 x_2 x_3 = 0$, where (x_1, x_2, x_3) denote the coordinates on $\mathbf{C}^3$*

Proof: Here we give only the proof of the assertion (v). For the proofs of the rest of the assertions, we refer to [T1] and [T4]. Let $X(n,k) := \mathbf{C}^n/\mu_k$, where μ_k is the cyclic group of k−th root of 1, acting by $\epsilon : (x_1, \cdots, x_n) \to (\epsilon x_1, \cdots, \epsilon x_n)$. Then the affine cone over the k−fold Veronese embedding of $\mathbf{P}^{n-1}(\mathbf{C})$ is isomorphic to $X(n,k)$. The map

$$\mathbf{C}^n \ni (x_1, \cdots, x_n) \to (x_1^k, \cdots, x_n^k) \in \mathbf{C}^n$$

factors through $X(n,k)$ and induces a quasi-ordinary projection $X(n,k) \to \mathbf{C}^n$. Since $(C_V, 0)$ is isomorphic to $X(3,2)$, it is quasi-ordinary. □

Hypersurface section: Let

$$\begin{aligned} \mathbf{C}^4 &\supset H : \ w = f(x,y,z), \\ \mathbf{C}^6 &\supset p^*H : \ x_0 + x_1 + x_2 + f(y_0, y_1, y_2) = 0, \end{aligned}$$

where f is a sufficiently *generic* holomorphic function defined in a small open neighborhood of the origin with $f(0,0,0) = 0$, and p is the linear projection $\mathbf{C}^6 \to \mathbf{C}^4$ induced by $\overline{p} : \mathbf{P}^5(\mathbf{C}) \to \mathbf{P}^3(\mathbf{C})$ as before. Let

$$\begin{aligned} T \cap H &: \ (xy)^2 + (yz)^2 + (zx)^2 + xyzf(x,y,z) = 0, \\ C_V \cap p^*H &: \ \text{the intersection of } C_V \text{ with } p^*H. \end{aligned}$$

Proposition 2.2. *$(C_V \cap p^*H, 0)$ is normal, and so $p_{|C_V \cap p^*H} : (C_V \cap p^*H, 0) \to (T \cap H, 0)$ gives the normalization of $(T \cap H, 0)$.*

Proof: Since $x_0 + x_1 + x_2 + f(y_0, y_1, y_2)$ is a non-zero divisor in $\mathcal{O}_{C_V,0}$, $Prof\, \mathcal{O}_{C_V \cap p^*H,0} = Prof\, \mathcal{O}_{C_V,0} - 1 = 2$. Hence $(C_V \cap p^*H, 0)$ is normal. □

Proposition 2.3. *$(T \cap H, 0)$ becomes a surface with only ordinary double points by the blowing-up at the origin, and a generic hypersurface section of $(T \cap H, 0)$ is an ordinary quadruple point of a curve.*

Proof: The tangent cone $C_0(T \cap H)$ to $T \cap H$ at the origin 0 of $\mathbf{C}^4$ is given by

$$(xy)^2 + (yz)^2 + (zx)^2 + xyz(ax + by + cz) = 0, \tag{2.4}$$

where a, b, c are sufficiently generic complex numbers. We denote by $\overline{C}$ the curve in $\mathbf{P}^2(\mathbf{C})$ defined by the equation (2.4). By Bertini's theorem, the curve

$\overline{C}$ is singular only at the three points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, if we take sufficiently *generic* complex numbers a, b, c. Furthermore, we may assume that these are ordinary double points. Therefore, since $C_0(T \cap H)$ is the cone over $\overline{C}$, the assertion follows. $\square$

Proposition 2.4. *$C_V \cap p^* C_0(H)$ is isomorphic to the cone over the twisted rational curve of degree* 4 *in* $\mathrm{P}^4(\mathbf{C})$.

Proof: Since the defining equation of $p^*C_0(H)$ in $\mathbf{C}^6$ is $x_0 + x_1 + x_2 + ay_0 + by_1 + cy_1 = 0$, the tangent cone $C_0(C_V \cap p^*H)$ to $C_V \cap p^*H$ at the origin 0 of $\mathbf{C}^6$ is given by

$$C_V \cap p^* C_0(H) = C_{V \cap \overline{p}^* \overline{C_0(H)}}, \tag{2.5}$$

where $\overline{C_0(H)}$ denotes the hyperplane in $\mathrm{P}^3(\mathbf{C})$ defined by $C_0(H)$. Note that $\overline{p}^*\overline{C_0(H)}$ is nothing but the hyperplane in $\mathrm{P}^5(\mathbf{C})$ defined by $p^*C_0(H)$. The pull-back of $V \cap \overline{C_0(H)}$ by the 2-fold Veronese embedding $v : \mathrm{P}^3(\mathbf{C}) \to \mathrm{P}^5(\mathbf{C})$ is

$$C_Q : Q(\xi_0, \xi_1, \xi_2) = \xi_0 + \xi_1 + \xi_2 + a\xi_0\xi_1 + b\xi_0\xi_2 + c\xi_1\xi_2 = 0.$$

We may assume that the quadric C_Q is non-singular, since the complex numbers a, b, c are sufficiently *generic*. Then there exist quadratic forms $p_0(s,t), p_1(s,t), p_2(s,t)$ of two variables s, t so that if we define the 2-fold Veronese map u from $\mathrm{P}^1(\mathbf{C})$ to $\mathrm{P}^2(\mathbf{C})$ by

$$(s:t) \in \mathrm{P}^1(\mathbf{C}) \mapsto (p_0(s,t) : p_1(s,t) : p_2(s,t)) = (\xi_0 : \xi_1 : \xi_2) \in \mathrm{P}^2(\mathbf{C}), \tag{2.6}$$

then the image of $\mathrm{P}^1(\mathbf{C})$ by the map u coincides with the quadric C_Q. The quadratic forms p_i $(0 \leq i \leq 2)$ satisfy

$$p_0^2 + p_1^2 + p_2^2 + ap_0p_1 + bp_0p_2 + cp_1p_2 = 0, \tag{2.7}$$

and the composite map $v \circ u : \mathrm{P}^1(\mathbf{C}) \to \mathrm{P}^5(\mathbf{C})$ gives rise to an isomorphism between $\mathrm{P}^1(\mathbf{C})$ and $V \cap \overline{p}^*\overline{C_0(H)}$ in $\mathrm{P}^5(\mathbf{C})$. The relation among p_ip_j's $(0 \leq i \leq j \leq 2)$ in (2.7) is the only one linear relation among p_ip_j's, because the existence of another linear relation among p_ip_j's linearly independent of (2.7) would contradict the fact that the map u in (2.6) is an embedding. Therefore $V \cap \overline{p}^*\overline{C_0(H)}$ is isomorphic to the image of $\mathrm{P}^1(\mathbf{C})$ into $\mathrm{P}^4(\mathbf{C})$ by the 4-fold Veronese map. Then we are done because of (2.5). $\square$

3 Lefschetz pencil on $\overline{X}$ and Euler number of the normalization of $\overline{X}$

Lefschetz pencil on $\overline{X}$: Throughout this section we denote by $\overline{X}$ an irreducible hypersurface in $P^4(\mathbf{C})$, which is defined locally at every point of $\overline{X}$ by one of the equations from (i) through (vi) in Proposition 1.1 with respect to a suitable local holomorphic coordinate system (x, y, z, w). We denote by $\overline{D}$ the double point locus (surface) of $\overline{X}$, i.e. the singular locus of $\overline{X}$, by $\overline{T}$ the triple point locus (curve) of $\overline{X}$, by $\overline{C}$ the cuspidal point locus (curve) of $\overline{X}$, and by $\Sigma\overline{q}$ the quadruple point locus of $\overline{X}$. Let P_∞ be a 2-dimensional linear subspace of $P^4(\mathbf{C})$ such that $\overline{C}_\infty := P_\infty \cap \overline{X}$ is an irreducible curve with ordinary double points in $P_\infty \simeq P^2(\mathbf{C})$. Let P be a 1-dimensional linear subspace of $P^4(\mathbf{C})$ situated in twisted position with respect to P_∞, i.e. the linear subspace $L(P_\infty, P)$ generated by P_∞ and P coincides with $P^4(\mathbf{C})$. Let $\pi : \overline{X} \setminus \overline{C}_\infty \to P$ be the linear projection with center $\overline{C}_\infty$, i.e., $\pi(x) := H_x \cap P$ for $x \in \overline{X} \setminus C_\infty$, where $H_x = L(x, P_\infty)$ is the hyperplane generated by x and P_∞. We put $\overline{X}_\lambda := H_\lambda \cap \overline{X}$ for $\lambda \in P$ and put $\overline{\mathcal{L}} : = \bigcup_{\lambda \in P} \overline{X}_\lambda$, which is a linear pencil on $\overline{X}$ with the base point locus $Bs(\overline{\mathcal{L}}) = \overline{C}_\infty$. Let $n_{\overline{X}} : X \to \overline{X}$ be the normalization map, and $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$ the pull-back of $\overline{\mathcal{L}}$ to X by $n_{\overline{X}}$. By use of the argument similar to that in [T2], we obtained the following:

Proposition 3.1. *If we take P_∞ sufficiently general, then there exists a finite set of points $Q := \{\lambda_1, \cdots, \lambda_q\}$ of P such that:*

(i) *$\overline{X}_{\lambda_i}$ contains only one quadruple point of $\overline{X}$, which we denote by $\overline{q}_{\lambda_i}$, and $X_{\lambda_i} := n_{\overline{X}}^{-1}(\overline{X}_{\lambda_i})$ is non-singular outside $n_{\overline{X}}^{-1}(\overline{q}_{\lambda_i})$ for any i with $1 \leq i \leq q$.*

(ii) *$\overline{X}_\lambda$ contains no quadruple point of $\overline{X}$ for any point $\lambda \in P \backslash Q$.*

(iii) *There exists a finite set of points $\{\mu_1, \cdots, \mu_c\}$ of $P \backslash Q$ such that:*

  (a) *$X_\lambda := n_{\overline{X}}^{-1}(\overline{X}_\lambda)$ is non-singular for $\lambda \in P \backslash Q$ with $\lambda \neq \mu_i$ $(1 \leq i \leq c)$, and*

  (b) *X_{μ_i} is a surface with only one isolated ordinary double point which is contained in $X \setminus n^{-1}(\overline{C}_\infty)$ for any i with $1 \leq i \leq c$, where c is the class (number) of $\overline{X}$ in $P^4(\mathbf{C})$, i.e. the degree of the top polar class $[M_3]$ of $\overline{X}$ in $P^4(\mathbf{C})$.*

In the sequel we assume that the linear pencil $\overline{\mathcal{L}} := \bigcup_{\lambda \in P} \overline{X}_\lambda$ is such as in Proposition 3.1. Then the linear pencil $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$, the pull-back of $\overline{\mathcal{L}} := \bigcup_{\lambda \in P} \overline{X}_\lambda$ to X by the normalization map $n_{\overline{X}} : X \to \overline{X}$, has $C_\infty := n^{-1}(\overline{C}_\infty)$ as its base point locus. Let $\sigma : \widehat{X} \to X$ be the blowing-up along C_∞, and $\widehat{\mathcal{L}} := \bigcup_{\lambda \in P} \widehat{X}_\lambda$, the proper inverse of $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$. Then $\widehat{\mathcal{L}}$ gives a fibering of $\widehat{X}$ over $P \simeq P^1(\mathbf{C})$. Therefore the Euler number $\chi(\widehat{X})$ of $\widehat{X}$ is given by

$$
\begin{aligned}
\chi(\widehat{X}) &= \chi(P^1(\mathbf{C}))\chi(\widehat{X}_\lambda) + \sum_{i=1}^{q}(\chi(\widehat{X}_{\lambda_i}) - \chi(\widehat{X}_\lambda)) + \sum_{j=1}^{c}(\chi(\widehat{X}_{\mu_j}) - \chi(\widehat{X}_\lambda)) \\
&= 2\chi(\widehat{X}_\lambda) - c + \sum_{i=1}^{q}(\chi(\widehat{X}_{\lambda_i}) - \chi(\widehat{X}_\lambda)) \\
&= 2\chi(X_\lambda) - c + \sum_{i=1}^{q}(\chi(X_{\lambda_i}) - \chi(X_\lambda)),
\end{aligned}
$$

where $\widehat{X}_\lambda$ and X_λ denote generic members of $\widehat{\mathcal{L}}$ and $\mathcal{L}$ respectively. Here the second equality above follows from the fact that a topological 2-cycle vanishes when $\lambda \to \mu_j$ for $j = 1, \cdots, c$. We put $\widehat{E} := \sigma^{-1}(C_\infty)$. Then, since $\widehat{X} \setminus \widehat{E} \simeq X \setminus C_\infty$,

$$
\begin{aligned}
\chi(\widehat{X}) - \chi(X) &= \chi(\widehat{E}) - \chi(C_\infty) \\
&= \chi(P^1(C))\chi(C_\infty) - \chi(C_\infty) \\
&= \chi(C_\infty)
\end{aligned}
$$

Hence,

$$
\chi(X) = 2\chi(X_\lambda) - \chi(C_\infty) - c + \sum_{i=1}^{q}(\chi(X_{\lambda_i}) - \chi(X_\lambda)). \qquad (3.1)
$$

Since X_λ is the normalization of a surface with ordinary singularities $\overline{X}_\lambda$ in $P^3(\mathbf{C})$, by the classical formula,

$$
\chi(X_\lambda) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma,
$$

and, since C_∞ is the normalization of the plane curve $\overline{C}_\infty$ whose degree is equal to n and has m ordinary double points,

$$
\chi(C_\infty) = 2 - 2g(C_\infty) = 2 - (n-1)(n-2) + 2m,
$$

where $n := deg\,\overline{X}$, $m := deg\,\overline{D}$, $t := deg\,\overline{T}$ and $\gamma := deg\,\overline{C}$.

Lefschetz pencil on $\overline{S}$: In the sequel we denote by $\overline{S}$ one of $\overline{X}_{\lambda_j}$, $(j = 1, \cdots, q)$, i.e. an irreducible hypersurface in $P^3(\mathbf{C})$ which is locally isomorphic to one of the following germs of two dimensional hypersurface singularities at the origin of $\mathbf{C}^3$ at every point of $\overline{S}$:

(i) $z = 0$ (simple point),
(ii) $yz = 0$ (ordinary double point),
(iii) $xyz = 0$ (ordinary triple point),
(iv) $xy^2 - z^2 = 0$ (cuspidal point),
(v) $(xy)^2 + (yz)^2 + (zx)^2 + xyz\phi(x, y, z) = 0$ (confluence of three ordinary double points),

where (x, y, z) are the coordinates on $\mathbf{C}^3$, and ϕ is a sufficiently *generic* holomorphic function defined in a small open neighborhood of the origin with $\phi(0, 0, 0) = 0$. Furthermore, $\overline{S}$ has the singularity (v) at just one point. We denote by $n_{\overline{S}} : S \to \overline{S}$ the normalization of $\overline{S}$. Similarly as in the case of $\overline{X}$, we have the following:

Proposition 3.2. *There exists a linear pencil of hyperplane sections* $\overline{\mathcal{L}}_S := \bigcup_{\lambda \in P} \overline{S}_\lambda$ $(P \simeq P^1(\mathbf{C}))$ *on* $\overline{S}$*, satisfying the following conditions:*

(i) *The base point locus* $Bs(\overline{\mathcal{L}}_S)$ *is* n *distinct points* $(n := deg\,\overline{X})$.

(ii) *There exists just one point* $\lambda_0 \in P$ *such that:*

  (a) $\overline{S}_{\lambda_0}$ *is a plane curve of degree* n*, having only one ordinary quadruple point, which we denote by* $\overline{q}_{\lambda_0}$*, and* $m-4$ *ordinary double points* $(m := deg\,\overline{D})$ *as singularities, and*

  (b) $S_{\lambda_0} := n_{\overline{S}}^{-1}(\overline{S}_{\lambda_0})$ *is non-singular outside* $n_{\overline{S}}^{-1}(\overline{q}_{\lambda_0})$.

(iii) *There exists a finite set of points* $\{\mu_1, \cdots, \mu_{c_{\overline{S}}}\}$ *of* $P - \{\lambda_0\}$ *such that:*

  (c) $S_\lambda := n_{\overline{S}}^{-1}(\overline{S}_\lambda)$ *is non-singular for any* $\lambda \in P - \{\lambda_0\}$ *with* $\lambda \neq \mu_i$ $(1 \leq i \leq c_{\overline{S}})$*, and*

  (d) S_{μ_i} $(1 \leq i \leq c_{\overline{S}})$ *is a curve with only one ordinary double point which is not contained in* $S \backslash n_{\overline{S}}^{-1}(Bs(\overline{\mathcal{L}}_S))$*, where* $c_{\overline{S}}$ *is the class (number) of* $\overline{S}$ *in* $P^3(\mathbf{C})$*, i.e. the degree of the top polar class* $[M_2]$ *of* $\overline{S}$ *in* $P^3(\mathbf{C})$.

By the same argument as in the case of $\overline{X}$, we have

$$\chi(S) = 2\chi(S_\lambda) + c_{\overline{S}} - n + (\chi(S_{\lambda_0}) - \chi(S_\lambda)), \tag{3.2}$$

where S_λ denotes a *generic* member of $\mathcal{L}_S := \bigcup_{\lambda\in P} S_\lambda$, the pull-back of $\overline{\mathcal{L}}_S := \bigcup_{\lambda\in P} \overline{S}_\lambda$ to S by the normalization map $n_{\overline{S}} : S \to \overline{S}$.

Lemma 3.3.

$$\chi(S_{\lambda_0}) - \chi(S_\lambda) = 1.$$

Proof: We denote by $n_0 : S^*_{\lambda_0} \to \overline{S}_{\lambda_0}$ the normalization of $\overline{S}_{\lambda_0}$, and by $\overline{q}_{\lambda_0}$ the quadruple point of $\overline{S}_{\lambda_0}$. Since $\overline{S}_{\lambda_0}$ is a plane curve of degree n with one ordinary quadruple point and $m-4$ ordinary double points, the genus $g(S^*_{\lambda_0})$ of the normalization $S^*_{\lambda_0}$ of $\overline{S}_{\lambda_0}$ is given by

$$g(S^*_{\lambda_0}) = \frac{1}{2}(n-1)(n-2) - m - 2. \tag{3.3}$$

Hence,

$$\chi(S^*_{\lambda_0}) = 2 - 2g(S^*_{\lambda_0}) = 2 - (n-1)(n-2) + 2(m+2). \tag{3.4}$$

Since S_{λ_0} is obtained by pushing forward the four distinct points $n_0^{-1}(\overline{q}_{\lambda_0})$ on $S^*_{\lambda_0}$ to the one point $n^{-1}_{\overline{S}|S_{\lambda_0}}(\overline{q}_{\lambda_0})$,

$$\chi(S^*_{\lambda_0}) - \chi(S_{\lambda_0}) = 3. \tag{3.5}$$

Hence, by (3.3), (3.4) and (3.5),

$$\chi(S_{\lambda_0}) = 2 - (n-1)(n-2) + 2m + 1. \tag{3.6}$$

On the other hand, since $\overline{S}_\lambda$ is a plane curve of degree n with m ordinary double points,

$$\chi(S_\lambda) = 2 - (n-1)(n-2) + 2m. \tag{3.7}$$

Therefore, by (3.6) and (3.7), we have $\chi(S_{\lambda_0}) - \chi(S_\lambda) = 1$. □

4 Calculation of Segre classes of singular subschemes

Throughout this section $\overline{X}$ and $\overline{S}$ are the same as in the previous section. In the sequel we will calculate the Segre classes of the singular subscheme of $\overline{X}$ in $P^4(\mathbf{C})$ (resp. $\overline{S}$ in $P^3(\mathbf{C})$) to know the *class* (number) c of $\overline{X}$ in $P^4(\mathbf{C})$ (resp. the *class* (number) $c_{\overline{S}}$ of $\overline{S}$ in $P^3(\mathbf{C})$). By Piene's formula ([P]), the polar classes of $\overline{X}$ (resp. $\overline{S}$) are described by use of its Segre classes. For the definition of Segre classes and their basic properties we refer to [T2].

Segre classes of the singular subscheme of $\overline{X}$ in $P^4(\mathbf{C})$: Throughout this section we fix the notation as follows:

$Y := P^4(\mathbf{C})$: the complex projective 4-space,
$\overline{J}$: the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$,
$\overline{D}$: the double point locus (surface) of $\overline{X}$, i.e. the singular locus of $\overline{X}$,
$\overline{T}$: the triple point locus (curve) of $\overline{X}$, which is equal to the singular locus of $\overline{D}$,
$\overline{C}$: the cuspidal point locus (curve) of $\overline{X}$,
$\Sigma\overline{q}$: the quadruple point locus of $\overline{X}$,
$n_{\overline{X}} : X \to \overline{X}$: the normalization of $\overline{X}$,
$f : X \to Y$: the composition of the normalization map $n_{\overline{X}}$ and the inclusion $\bar{\iota} : \overline{X} \hookrightarrow Y$,
J : the scheme-theoretic inverse of $\overline{J}$ by f,
D, T and C: the inverse images of $\overline{D}$, $\overline{T}$ and C by f, respectively,
Σq : the inverse image of $\Sigma\overline{q}$ by f.

We consider the following diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \sigma} \\ X & \xrightarrow[f]{} & Y, \end{array} \quad . \tag{4.1}$$

where:

$\sigma : Y' \to Y$: the blowing-up of Y along the quadruple point locus $\Sigma\overline{q}$ of $\overline{X}$,
$\tau : X' \to X$: the blowing-up of X along Σq,
$f' : X' \to Y'$: the map which makes the diagram above commute.

We put

$\overline{X}'$: the proper inverse image of $\overline{X}$ by σ, which is nothing but $f'(X')$,

$\overline{J}'$: the singular subscheme of $\overline{X}'$ defined by the Jacobian ideal of $\overline{X}'$,
$\overline{D}'$, $\overline{T}'$ and $\overline{C}'$: the proper inverse images of $\overline{D}$, $\overline{T}$ and $\overline{C}'$ by σ, respectively,
$\overline{E} := \sigma^{-1}(\Sigma\overline{q}) = \Sigma_{\overline{q}}E_{\overline{q}}$: the exceptional divisor of the blowing-up σ, where $\overline{E}_q$ denotes the exceptional divisor corresponding to a quadruple point $\overline{q}$,
D', T' and C': the inverse images of D, T and C by τ, respectively, which are nothing but the inverse images of $\overline{D}'$, $\overline{T}'$ and $\overline{C}'$ by f', respectively.

Note that $\overline{X}'$ is a threefold with ordinary singularities. We denote by $\overline{J}''$ the scheme-theoretic inverse of $\overline{J}$ by the map $\sigma_{|\overline{X}'} : \overline{X}' \to \overline{X}$, the restriction of σ to $\overline{X}'$. Calculating directly by use of local coordinates, we have

$$\overline{J}'' = \overline{J}' + 3\overline{X}' \cdot \overline{E}.$$

We denote by J'' the scheme-theoretic inverse of $\overline{J}''$by the map f'. Then we have

$$J'' = D' + 3E + C',$$

which also comes from a direct calculation, using the concrete description of the map f' in terms of local coordinates. If we put $D'' := D' + 3E$, then by the formula (3.1) in [T2] (p. 284), we have the following equalities concerning the Segre classes of J'' in X':

$$\left\{ \begin{array}{l} s(J'', X')_2 = [D''] \\ s(J'', X')_1 = -[D'']^2 + [C'] \\ s(J'', X')_0 = [D'']^3 - c_1(N_{C'/X'}) \cap [C'] - 3D'' \cdot C' \end{array} \right. \tag{4.2}$$

Since

$$\begin{array}{l} [D'']^2 = [D']^2 + 6D' \cdot E + 9[E]^2, \\ [D'']^3 = [D']^3 + 9[D']^2 \cdot E + 27D' \cdot [E]^2 + 27[E]^3, \\ D'' \cdot C' = D' \cdot C' + 3E \cdot C', \end{array}$$

and since

$$\left\{ \begin{array}{l} f'_*[D']^2 = \overline{X}' \cdot \overline{D}' + 3\overline{T}' - \overline{C}', \\ f'_*[D']^3 = [\overline{X}']^2 \cdot [\overline{D}'] - 2[\overline{D}']^2 + 5\overline{X}' \cdot \overline{T}' - \overline{X}' \cdot \overline{C}', \\ f'_*(c_1(N_{C'/X'}) \cap [C']) = -K_{Y'} \cdot \overline{C}' - \overline{X}' \cdot \overline{C}' + k_{\overline{C}'}, \\ f'_*(D' \cdot C') = 0, \end{array} \right.$$

which are the results in [T2], pushing forward the Segre classes $s(J'', X')_i$ ($0 \leq i \leq 2$) in (4.2), we have the following:

Proposition 4.1. *The Segre classes of the subscheme $\overline{J}''$ in $\overline{X}'$ are given as follows:*

$$\text{(i)}\qquad s(\overline{J}'',\overline{X}')_2 = 2\overline{D}' + 3\overline{X}'\cdot\overline{E},$$

$$\text{(ii)}\qquad s(\overline{J}'',\overline{X}')_1 = -\overline{X}'\cdot\overline{D}' - 3\overline{T}' + 2\overline{C}' - 12\overline{D}'\cdot\overline{E} - 9\overline{X}'\cdot[\overline{E}]^2,$$

$$\begin{aligned}\text{(iii)}\qquad s(\overline{J}'',\overline{X}')_0 &= [\overline{X}']^2\cdot\overline{D}' - 2[\overline{D}']^2 + 5\overline{X}'\cdot\overline{T}' + K_{Y'}\cdot\overline{C}' - [k_{\overline{C}'}]\\ &+9\overline{X}'\cdot\overline{D}'\cdot\overline{E} + 27\overline{T}'\cdot\overline{E} - 18\overline{C}'\cdot\overline{E} + 54\overline{D}'\cdot[\overline{E}]^2 + 27\overline{X}'\cdot[\overline{E}]^3,\end{aligned}$$

where $K_{Y'}$ is the canonical divisor of Y', and $k_{\overline{C}'}$ is that of $\overline{C}'$.

Lemma 4.2.

(i) $\overline{X}'\cdot\overline{E} = 4j_*(\Sigma_{\overline{q}}[H_{\overline{q}}])$, (ii) $\overline{D}'\cdot\overline{E} = 3j_*(\Sigma_{\overline{q}}[H_{\overline{q}}]^2)$,

(iii) $\overline{T}'\cdot\overline{E} = j_*(\Sigma_{\overline{q}}[H_{\overline{q}}]^3)$, (iv) $\overline{C}'\cdot\overline{E} = 6j_*(\Sigma_{\overline{q}}[H_{\overline{q}}]^3)$,

where $H_{\overline{q}}$ denotes a hyperplane in $E_{\overline{q}}$, the exceptional divisor corresponding to a quadruple point $\overline{q}$ of $\overline{X}$, and $j : \overline{E} \hookrightarrow Y'$ the inclusion map.

Proof: $\overline{X}$ is locally isomorphic to the cone over the Steiner surface $\overline{S}$ at a quadruple point $\overline{q}$, and so the assertions follows. □

Since the multiplicity of $\overline{X}$ at each quadruple point $\overline{q}$ is four, we have

$$\sigma^*[\overline{X}] = \overline{X}' + 4\overline{E},$$

and since the multiplicity of $\overline{D}$ at each quadruple point $\overline{q}$ is 3, by the *blow-up formula* ([F], Theorem 6.7, p.116 and Corollary 6.7, p.117), we have

$$\sigma^*[\overline{D}] = \overline{D}' + 3j_*[\Sigma_{\overline{q}}H_{\overline{q}}].$$

Calculating push-forward of the Segre classes $s(\overline{J}'',\overline{X}')_i$ ($0\le i\le 2$) in Proposition 4.1 by σ, using the facts above and Lemma 4.2, we have the following:

Proposition 4.3. *The Segre classes of the singular subscheme $\overline{J}$ of $\overline{X}$ are given as follows:*

$$\text{(i)}\qquad s(\overline{J},\overline{X})_2 = 2\overline{D},$$

$$\text{(ii)}\qquad s(\overline{J},\overline{X})_1 = -\overline{X}\cdot\overline{D} - 3\overline{T} + 2\overline{C},$$

(iii) $$s(\overline{J},\overline{X})_0 = [\overline{X}]^2 \cdot \overline{D} - 2[\overline{D}]^2 + 5\overline{X} \cdot \overline{T} + K_Y \cdot \overline{C} - \sigma_*[k_{\overline{C}'}] - 59[\Sigma\overline{q}],$$

where K_Y is the canonical divisor of Y, and $\sigma_*[k_{\overline{C}'}]$ is the direct image of the canonical divisor $k_{\overline{C}'}$ of $\overline{C}'$ by the map $\sigma_{|\overline{C}'} : \overline{C}' \to \overline{C}$.

Corollary 4.4. *Let $\overline{X}_0$ be a hypersurface in $P^4(\mathbf{C})$ whose degrees of the various singular loci are the same as those of $\overline{X}$ we are considering in this article, but without quadruple points. Then:*

$$c - c_0 = deg\,[k_{\overline{C}'}] - deg\,[k_{\overline{C}_0}] + 59\#[\Sigma\overline{q}],$$

where c (resp. c_0) denotes the class (number) of $\overline{X}$ (resp. $\overline{X}_0$) in $P^4(\mathbf{C})$, $\overline{C}_0$ the cuspidal point locus (curve) of $\overline{X}_0$, and $k_{\overline{C}_0}$ the canonical divisor of $\overline{C}_0$.

Proof: By Piene's formula,

$$c = (n-1)^3 deg\,\overline{X} - 3(n-1)^2 deg\,s_2 - 3(n-1) deg\,s_1 - deg\,s_0$$

Hence, by Proposition 4.3 above and Proposition 3.6 in [T2], the assertion follows. □

<u>Segre classes of the singular subscheme of $\overline{S}$ in $P^3(\mathbf{C})$:</u> To calculate the Segre classes of the singular subscheme of $\overline{S}$, we consider the following diagram instead of the diagram (4.1):

$$\begin{array}{ccc} S' & \xrightarrow{g'} & Z' \\ \tau_S \downarrow & & \downarrow \sigma_S \\ S & \xrightarrow[g]{} & Z, \end{array} \qquad (4.3)$$

where:
$Z := P^3(\mathbf{C})$,
$g : S \to Z$: the composite of the normalization map $n_S : S \to \overline{S}$ and the inclusion $\overline{\iota} : \overline{S} \hookrightarrow Z$,
$\sigma_S : Z' \to Z$: the blowing-up of Z at the quadruple point $\overline{q}$ of $\overline{S}$,
$\tau_S : S' \to S$: the blowing-up of S at $q := g^{-1}(\overline{q})$,
$g' : S' \to Z'$: the map which makes the diagram (4.3) commute.
We put
$\overline{D}_S$: the double point locus (curve) of $\overline{S}$, i.e. the singular locus of $\overline{S}$,
$\Sigma\overline{t}$: the triple point locus of $\overline{S}$, which is equal to the singular locus of $\overline{D}_S$,

$\Sigma\overline{c}$: the cuspidal point locus of $\overline{S}$,
$\overline{S}'$: the proper inverse image of $\overline{S}$ by σ_S, which is nothing but $g'(S')$,
$\overline{D}'_S$, $\Sigma\overline{t}'$ and $\Sigma\overline{c}'$: the proper inverse images of $\overline{D}_S$, $\Sigma\overline{t}$ and $\Sigma\overline{c}$ by σ_S, respectively,
D_S, Σt and Σc: the proper inverse images of $\overline{D}_S$, $\Sigma\overline{t}$ and $\Sigma\overline{c}$ by g, respectively,
D'_S, $\Sigma t'$ and $\Sigma c'$: the proper inverse images of D_S, Σt and Σc by τ_S, respectively, which is nothing but the proper inverse images of $\overline{D}'_S$, $\Sigma\overline{t}'$ and $\Sigma\overline{c}'$ by g', respectively.
$\overline{J}$, $\overline{J}'$, J, J', $\overline{E}$ and E for $\overline{S}$ are similarly defined as in the case of $\overline{X}$.

Note that $\overline{S}'$ is an algebraic surface with ordinary singularities. In the sequel we assume that $\overline{S}$ is defined by the equation in (2.4) at each quadruple point of $\overline{S}$. We are allowed to do this, because the Segre classes of a singular subscheme depend only on its tangent cone. We denote by $\overline{J}''$ the scheme-theoretic inverse of $\overline{J}$ by the map $\sigma_{S|\overline{S}'} : \overline{S}' \to \overline{S}$, the restriction of σ_S to $\overline{S}'$. Calculating directly by use of local coordinates, we have

$$\overline{J}'' = \overline{J}' + 3\overline{S}' \cdot \overline{E}.$$

We denote by J'' the scheme-theoretic inverse of $\overline{J}''$by the map g'. Then we have

$$J'' = D'_S + 3E + \Sigma c',$$

which also comes from a direct calculation, using the concrete description of the map g' in terms of local coordinates. If we put $D''_S := D'_S + 3E$, then by Proposition 3.2 in [T2] (p. 284) ([F], Proposition 9.2, p.161), we have the following equalities concerning the Segre classes of J'' in S':

$$\begin{cases} s(J'', S')_1 = [D''_S] \\ s(J'', S')_0 = -[D''_S]^2 + [\Sigma c'] \end{cases} \tag{4.4}$$

Proposition 4.5. *The Segre classes of the subscheme $\overline{J}''$ in $\overline{S}'$ are given as follows:*

(i) $\quad s(\overline{J}'', \overline{S}')_1 = 2[\overline{D}'_S] + 3\overline{S}' \cdot \overline{E},$

(ii) $\quad s(\overline{J}'', \overline{S}')_0 = -\overline{S}' \cdot \overline{D}'_S - 3[\Sigma\overline{t}'] + 2[\Sigma\overline{c}'] - 12\overline{D}'_S \cdot \overline{E} - 9\overline{S}' \cdot [\overline{E}]^2.$

Proof:

$$\begin{aligned} s(\overline{J}',\overline{S}')_1 &= g'_* s(\overline{J}'',\overline{S}')_1 \\ &= g'_*[D''_s] = g'_*[D'_s] + 3g'_*[E] \\ &= 2[\overline{D}'_S] + 3\overline{S}' \cdot \overline{E}. \end{aligned}$$

This proves the assertion (i). Since $\overline{S}'$ is an algebraic surface with ordinary singularities, we have

$$g'_*[D'_s]^2 = \overline{S}' \cdot \overline{D}'_S - [\Sigma\overline{c}'_S] + 3[\Sigma\overline{t}'_S]$$

(cf. [F], Example 9.3.7, p.168). Hence

$$\begin{aligned} g'_*[D''_s]^2 &= g'_*[D'_S + 3E]^2 \\ &= g'_*[D'_s]^2 + 6g'_*[D'_S] + 9g'_*[E]^2 \\ &= \overline{S}' \cdot \overline{D}'_S - [\Sigma\overline{c}'_S] + 3[\Sigma\overline{t}'_S] + 12\overline{D}'_S \cdot \overline{E} + 9\overline{S}' \cdot [\overline{E}]^2. \end{aligned}$$

Therefore, calculating push-forward of $s(\overline{J}'',\overline{S}')_0$ in (4.4) by g', we have the assertion (ii). □

Lemma 4.6.

(i) $\overline{S}' \cdot \overline{E} = 4k_*[\overline{H}]$, (ii) $\overline{D}'_S \cdot \overline{E} = 3k_*[\overline{H}]^2$,

where $\overline{H}$ denotes a hyperplane in $\overline{E}$, the exceptional divisor corresponding to the quadruple point $\overline{q}$ of $\overline{S}$, and $k : \overline{E} \hookrightarrow Z'$ the inclusion map.

Proof: $\overline{S}$ can be considered to be locally isomorphic to the cone over a plane curve $\overline{C}$ of degree four which has three ordinary double points at the quadruple point $\overline{q}$ (cf. Proposition 2.3), and so the assertions follow. □

Since the multiplicity of $\overline{S}$ at the quadruple point $\overline{q}$ is four, we have

$$\sigma^*[\overline{S}] = \overline{S}' + 4\overline{E}.$$

Calculating push-forward of the Segre classes $s(\overline{J}',\overline{S}')_i$ ($0 \leq i \leq 1$) in Proposition 4.5 by σ, using the fact above and Lemma 4.6, we have the following:

Proposition 4.7. *The Segre classes of the singular subscheme $\overline{J}$ of $\overline{S}$ are given as follows:*

(i) $s(\overline{J}, \overline{S})_1 = 2\overline{D}_S$,

(ii) $s(\overline{J}, \overline{S})_0 = -\overline{S} \cdot \overline{D}_S - 3[\Sigma \overline{t}] + 2[\Sigma \overline{c}] + 12[\overline{q}]$.

Corollary 4.8. *The effect of the existence of the quadruple point $\overline{q}$ to the class (number) $c_{\overline{S}}$ of $\overline{S}$ in $P^3(\mathrm{C})$ is -12.*

Proof: By Piene's formula,

$$c_{\overline{S}} = (n-1)^2 deg\,\overline{S} - 2(n-1) deg\, s_1 - deg\, s_0$$

Therefore, by Proposition 4.7 the assertion follows. □

Corollary 4.9. *Let X_{λ_i} be a member of the linear system $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$ for which $\overline{X}_{\lambda_i}$ containing a quadruple point of $\overline{X}$, and X_λ a generic member of $\mathcal{L}$. Then*

$$\chi(X_{\lambda_i}) - \chi(X_\lambda) = -11$$

Proof: We set $S := X_{\lambda_i}$, $\overline{S} := \overline{X}_{\lambda_i}$, $S_0 := X_\lambda$ and $\overline{S}_0 := \overline{X}_\lambda$. Then by (3.2),

$$\chi(S) - \chi(S_0) = c_{\overline{S}} - c_{\overline{S}_0} + \chi(S_{\lambda_0}) - \chi(S_\lambda),$$

where $c_{\overline{S}}$, $c_{\overline{S}_0}$ are the class numbers of $\overline{S}$ and $\overline{S}_0$ in $P^3(\mathrm{C})$, respectively, S_{λ_0} is the member of the linear pencil $\mathcal{L}_S := \bigcup_{\lambda \in P} S_\lambda$ for which $\overline{S}_{\lambda_0}$ containing the quadruple point of $\overline{S}$, and S_λ is a generic member of $\mathcal{L}_S$. Therefore, by Corollary 4.8 and Lemma 3.3, we obtain the assertion. □

5 A conclusion

Theorem 5.1. *Let $\overline{X}_0$ be a hypersurface in $P^4(\mathrm{C})$ whose degrees of the various singular loci are the same as those of $\overline{X}$ we are considering in this article, but without quadruple points, and let X_0 be the normal model of $\overline{X}_0$. Then:*

$$\chi(X) - \chi(X_0) = deg\, k_{\overline{C}_0} - deg\, k_{\overline{C}'} - 70\#[\Sigma \overline{q}],$$

where $k_{\overline{C}_0}$ is the canonical divisor of the cuspidal point locus (curve) $\overline{C}_0$ of $\overline{X}_0$, and $k_{\overline{C}'}$ is that of the normal model $\overline{C}'$ of the cuspidal point locus (curve) $\overline{C}$ of $\overline{X}$.

Proof: By (3.1), Corollary 4.4 and Corollary 4.9,

$$\begin{aligned}\chi(X)-\chi(X_0) &= -(c-c_0)+\Sigma_{i=1}^{q}(\chi(X_{\lambda_i})-\chi(X_\lambda)) \\ &= -(deg\, k_{\overline{C}'}-deg\, k_{\overline{C}_0}+59\#[\Sigma\overline{q}])-11\#[\Sigma\overline{q}]\end{aligned}$$

where c and c_0 are the *class numbers* of $\overline{X}$ and $\overline{X}_0$ in $P^4(\mathbf{C})$ respectively. □

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