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THE EULER NUMBER OF THE NORMALIZATION OF AN ALGEBRAIC THREEFOLD WITH ORDINARY SINGULARITIES

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Abstract. By a classical formula due to Enriques, the Euler number $\chi(X)$ of the non-singular normalization X of an algebraic surface S with ordinary singularities in $P^3(\mathbf{C})$ is given by $\chi(X) = n(n^2-4n+6)-(3n-8)m+3t-2\gamma$, where $n =$ the degree of S, $m =$ the degree of the double curve (singular locus) D_S of S, $t =$ the cardinal number of the triple points of S, and γ=the cardinal number of the cuspidal points of S. In this article we shall give a similar formula for an algebraic threefold with ordinary singularities in $P^4(\mathbf{C})$ which is free from quadruple points (Theorem 4.1).

1. Preliminaries. We begin with recalling some definitions.

DEFINITION 1. ([1]) An irreducible hypersurface S in the complex projective 3-space $P^3(\mathbf{C})$ is called an *algebraic surface with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 3-space $\mathbf{C}^3$ at every point of S:

$$\begin{cases} (i)\ z=0\ \text{(simple point)} & (ii)\ yz=0\ \text{(ordinary double point)} \\ (iii)\ xyz=0\ \text{(ordinary triple point)} & (iv)\ xy^2-z^2=0\ \text{(cuspidal point)}, \end{cases}$$

where (x, y, z) is the coordinate on $\mathbf{C}^3$.

DEFINITION 2. ([6]) An irreducible hypersurface T in the complex projective 4-space $P^4(\mathbf{C})$ is called an *algebraic threehold with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbf{C}^4$ at

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every point of T:

$$\begin{cases} (i)\ w=0\ \text{(simple point)} & (ii)\ zw=0\ \text{(ordinary double point)} \\ (iii)\ yzw=0\ \text{(ordinary triple point)} & (iv)\ xyzw=0\ \text{(ordinary quadruple point)} \\ (v)\ xy^2-z^2=0\ \text{(cuspidal point)} & (vi)\ w(xy^2-z^2)=0\ \text{(stationary point)}, \end{cases}$$

where (x,y,z,w) is the coordinate on $\mathbf{C}^4$.

It is known that every complex projective surface (*resp.* threefold) is birationally equivalent to an algebraic surface (*resp.* threefold) with ordinary singularities.

Next we give the definition of the *polar classes* of an r-dimensional subvariety X^r in a complex projective space $P^n(\mathbf{C})$. Denote by U the open subset of X^r consisting of all simple points of X. For a given linear $(n-r+k-2)$ dimensional subspace $L_{(k)}$ of $P^n(\mathbf{C})$, we let $M_k(U)$ denote the locus of points $x\in U$ such that the tangent space T_xX of X at x intersects $L_{(k)}$ in a space at least $k-1$ dimension.

DEFINITION 3. The closure M_k of $M_k(U)$ in X is called the *k-th polar locus* of X.

M_k has a natural reduced scheme structure and, for a general $L_{(k)}$, M_k has codimension k in X. Moreover, for such $L_{(k)}$, the rational equivalent class of the cycle defined by M_k does not depend on $L_{(k)}$ (cf. [5]).

DEFINITION 4. This class denoted by $[M_k]$ is called the *k-th polar class* of X. The degree μ_k of M_k is called the *k-th class*. The top class μ_r is called the *class* of X.

Now we give the definition of the *Segre class* of a closed subscheme X of a scheme Y. We denote by $\mathcal{I}$ the ideal sheaf of X in Y and put

$$S^{\cdot} := \Sigma_{k=0}^{\infty}\mathcal{I}^k/\mathcal{I}^{k+1},$$

which is a graded sheaf of $\mathcal{O}_X$-algebras on X. To $S^{\cdot}$ we associate two schemes over X: the *cone* of $S^{\cdot}$

$$C := Spec(S^{\cdot}), \qquad \pi : C \to X;$$

and the *projective cone* $P(C)$ to X in Y by

$$P(C) := Proj(S^{\cdot}), \qquad p : P(C) \to X.$$

C is called the *normal cone* to X in Y, denoted by C_XY , and $P(C)$ the *projective normal cone* to X in Y. On $P(C)$ there is a canonincal line bundle, denoted by $\mathcal{O}_C(1)$. Let z be a variable, $S^{\cdot}[z]$ the graded algebra whose n^{th} graded piece $(S^{\cdot}[z])^n$ is

$$S^n \oplus S^{n-1}z \oplus \cdots \oplus S^1z^{n-1} \oplus S^0z^n.$$

The corresponding cone is denoted by $C\oplus 1$. The cone

$$P(C\oplus 1) = Proj(S^{\cdot}[z]), \qquad q : P(C\oplus 1) \to X$$

is called the *projective completion* of C. The element z in $(S^{\cdot}[z])^1$ determines a regular section of $\mathcal{O}_{C\oplus 1}(1)$ on $P(C\oplus 1)$ whose zero-scheme is canonically isomorphic to $P(C)$. The complement to $P(C)$ in $P(C\oplus 1)$ is canonically isomorphic to C.

DEFINITION 5. The *Segre class* of X in Y, denoted by $s(X,Y)$, is the class in the graded Chow group A_*X of X defined by the formula

$$s(X,Y) := q_*(\Sigma_{i\geq 0}\, c_1(\mathcal{O}_{C\oplus 1}(1))^i \cap [P(C\oplus 1)]).$$

Note that $s(X,Y)$ is a birational invariant, which means that if $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, then the Segre class of X' in Y' pushes forward to the Segre class of X in Y. If the normal cone C_XY is a vector bundle N, then $s(X,Y) = c(N)^{-1} \cap [X]$ where $c(N)^{-1}$ denotes the total inverse Chern class of N (cf. [2], Chapter 4).

Finally, we give the definitions of *regular embeddings* and *local complete intersection morphisms* of schemes.

DEFINITION 6. We say a closed embedding $\iota : X \to Y$ of schemes is a *regular embedding of codimension* d if every point in X has an affine neighborhood U in Y, such that if A is the coordinate ring of U, I the ideal of A defining X, then I is generated by a regular sequence of length d.

If this is the case, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of X in Y, is a locally free sheaf of rank d. The *normal bundle* to X in Y, denoted by N_XY, is the vector bundle on X whose shef of sections is dual to $\mathcal{I}/\mathcal{I}^2$. Note that the normal bundle N_XY is canonically isomorphic to the normal cone C_XY for a (closed) regular embedding $\iota : X \to Y$ since the canonical map from $Sym(\mathcal{I}/\mathcal{I}^2)$ to $S^{\cdot} := \Sigma_{k=0}^{\infty}\mathcal{I}^k/\mathcal{I}^{k+1}$ is an isomorphism (cf. [2], Appendix B, B.7).

DEFINITION 7. A morphism $f : X \to Y$ is called a *local complete intersection morphism of codimension* d if f factors into a (closed) regular embedding $\iota : X \to Y$ of some constant codimension e, followed by a smooth morphism $p : P \to Y$ of constant relative dimension $d + e$.

2. The existence of a good linear pencil of hyperplane sections. Throughout this section we denote by X an algebraic threefold with ordinary singularities of degree n in the complex projective 4-space $P^4(\mathbf{C})$, by D the double surface of X, i.e., the singular locus of X, by T the triple points locus of X, by C the cuspidal point locus of X, by Σs the stationary point locus of X . Let m, t, γ be the degrees of D, T, C, respectively. Let P_∞ be a 2-dimensional linear subspace of $P^4(\mathbf{C})$ such that $C_\infty := P_\infty \cap X$ is an irreducible curve with ordinary double points in $P_\infty \simeq P^2(\mathbf{C})$. Let P be a 1-dimensional linear subspace of $P^4(\mathbf{C})$ situated in twisted position with respect to P_∞, i.e., the linear subspace $L(P_\infty, P)$ generated by P_∞ and P is equal to $P^4(\mathbf{C})$. Let $\pi : X \setminus C_\infty \to P$ be the linear projection with center C_∞, i.e., $\pi(x) := H_x \cap P$ for $x \in X \setminus C_\infty$, where $H_x = L(x, P_\infty)$ is the hyperplane generated by x and P_∞. We put $X_\lambda := H_\lambda \cap X$ for $\lambda \in P$ and $\mathcal{L} := \bigcup_{\lambda\in P} X_\lambda$. Then $\mathcal{L}$ is a linear system on X with the base point locus $Bs(\mathcal{L}) = C_\infty$. Let $f : X_1 \to X$ be the normalization map and $\widetilde{\mathcal{L}} := \bigcup_{\lambda\in P} \widetilde{X_\lambda}$ the pull-back of $\mathcal{L}$ to X_1.

THEOREM 2.1 *If we take P_∞ sufficiently general, there exists a finite set $\{\lambda_1, \cdots, \lambda_c\}$ of points of P such that*

(i) $\widetilde{X_\lambda}$ *is non-singular for* λ *with* $\lambda \neq \lambda_i$ $(1\underline{<}i\underline{<}c)$, *and*

(ii) $\widetilde{X_{\lambda_i}}$ *is a surface with only one isolated ordinary double point which is contained in* $X_1 \setminus f^{-1}(C_\infty)$ *for any* i *with* $1 \underline{<} i \underline{<} c$,

where c *is the class of* X.

PROOF. We consider the *Gauss map*

$$\Phi : X - - > P^4(C)^\vee$$

defined by

$$\Phi(p) = \Big[\frac{\partial F}{\partial x_0}(p) : \frac{\partial F}{\partial x_1}(p) : \frac{\partial F}{\partial x_2}(p) : \frac{\partial F}{\partial x_3}(p) : \frac{\partial F}{\partial x_4}(p)\Big] \tag{1}$$

for $p \in X$, where F is the homogeneous polynomial defining X in $P^4(C)$, $[x_0 : x_1 : x_2 : x_3 : x_4]$ the homogeneous coordinate on $P^4(\mathbf{C})$, and $P^4(\mathbf{C})^\vee$ the dual projective space of $P^4(\mathbf{C})$. Φ is a rational map, which is not defined on the singular locus D of X. Let $\overline{X}$ be the closure in $X \times P^4(\mathbf{C})^\vee$ of the graph of Φ. We denote by $\pi_1 : \overline{X} \to X$ the morphism induced by the projection to the first factor, and $\pi_2 : \overline{X} \to P^4(\mathbf{C})^\vee$ the one induced by the projection to the second factor. We call $\pi_1 : \overline{X} \to X$ the *Nash blow-up* of X. Note that the rational map Φ can be extended to $\overline{X}$ and $\overline{X}$ is minimal among the varieties with such property. In our case, since X is a hypersurface, $\overline{X}$ coincides with the the blow-up of the Jacobian ideal of X ([4], Remark 2, p.300). We denote by $X^\vee$ the image of $\overline{X}$ by $\pi_2 : \overline{X} \to P^4(\mathbf{C})^\vee$, and call it the *dual variety* of X. The dimension of $X^\vee$ is not less than 1, nor greater than 3 ([3], Example 15.22., p.196).

We are now going to define an algebraic subset B in $P^4(\mathbf{C})^\vee$, whose points correspond to hyperplanes in $P^4(\mathbf{C})$ being in *bad* positions in some sense at their intersecting points with the cuspidal point locus C, or stationary point locus Σs of X. Let p be a point of C, or Σs. Then there is an open neighborhood U of p and a complex analytic local coordinats (x, y, z, w) with center p such that the defining equation of X is given by one of the following:

$$xy^2 - z^2 = 0 \tag{2}$$

$$w(xy^2 - z^2) = 0. \tag{3}$$

Let $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ be a linear affine coordinate with center p, and H a hyperplane passing through p, defined by the equation

$$\sum_{i=1}^{i=4} a_i \zeta_i = 0 \quad (a_i \in \mathbf{C},\ 1 \underline{<} i \underline{<} 4). \tag{4}$$

We say H is in a *bad* position at the point p, if the coefficients of the equation (4) satisfy the following two conditions:

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) = 0, \tag{5}$$

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) = 0. \tag{6}$$

We define B_p to be the algebraic subset of $P^4(C)^\vee$ consisting of all points which corresponds to hyperplanes in $P^4(C)$ passing through p and being in a bad position at p in the sense defined above. We define an algebraic subset B of $P^4(C)^\vee$ by

$$B := \bigcup_{p \in C} B_p \tag{7}$$

Here we should note that the stationary points are included in C, and since $\dim B_p = 1$, the codimension of B is greater than 1. We choose a line L^* in $P^4(\mathbf{C})^\vee$ which satisfies all of the following conditions:

$$L^* \cap \{X^\vee \setminus \Phi(X_{sm})\} = \emptyset, \tag{8}$$

$$L^* \cap (X^\vee)_{sing} = \emptyset, \tag{9}$$

$$L^* \cap B = \emptyset, \tag{10}$$

$$L^* \text{ intersects transversely with } \Phi(X_{sm}) \setminus (X^\vee)_{sing}, \tag{11}$$

where X_{sm} denotes $X \setminus D$, the simple point locus of X, and $(X^\vee)_{sing}$ the singular point locus of $X^\vee$. This is always possible because all the codimensions of $X^\vee \setminus \Phi(X_{sm})$, $(X^\vee)_{sing}$ and B are greater than 1 in $P^4(\mathbf{C})^\vee$. Note that the cardinal number of the set $L^* \cap \{\Phi(X_{sm}) \setminus (X^\vee)_{sing}\}$ is nothing but the *class* of X. We denote by H_λ the hyperplane in $P^4(\mathbf{C})$ corresponding to each $\lambda \in L^*$. We put $X_\lambda := X \cap H_\lambda$ and consider the linear pencil

$$\mathcal{L} = \bigcup_{\lambda \in L^*} X_\lambda$$

of hyperplane sections of X. We are now going to show that the assertions (i) and (ii) of the proposition hold for the pull-back $\widetilde{\mathcal{L}} = \bigcup_{\lambda \in L^*} \widetilde{X_\lambda}$ of $\mathcal{L}$ to the normal model X_1 of X by the normalization map $f : X_1 \to X$.

The assertion (i): Let $\{\lambda_1, \cdots, \lambda_c\}$ be all of the distinct points of $L^* \cap \{\Phi(X_{sm}) \setminus (X^\vee)_{sing}\}$, and λ a point L^* with $\lambda \neq \lambda_i$ ($1 \underline{<} i \underline{<} c$). Then $\lambda \notin X^\vee$. This means that H_λ is not tangent to X at any point of X_{sm}, and not a limit of tangent hyperplanes to X_{sm}. Hence we infer that $\widetilde{X_\lambda}$ is non-singular at every point of $X_1 \setminus f^{-1}(C)$. Therefore what we have to do is to show that $\widetilde{X_\lambda}$ is non-singular at $f^{-1}(p)$ for any point $p \in H_\lambda \cap C$. In the subsequence we shall show this fact only when p is a statinary point, since the proof for a cuspidal point is more easy. Assume p is a cuspidal point of X and $p \in H_\lambda$. We take a complex analytic local coordinate (x, y, x, w) with center p such that the defining equation of X is given by the equation (3). We also take a linear affine coordinate $(\zeta_0, \zeta_1, \zeta_2, \zeta_3)$ with center p and assume that the defining equation of H_λ is given by the equation (4). We rewrite the equation (4) as

$$Ax + By + Cz + Dw = 0, \tag{12}$$

where A, B, C and D are complex analytic functions defined in a neighborhood of p. $f^{-1}(p)$ is two points, say q_1, q_2, where the normalization map $f : X_1 \to X$ is given as follows:

$$f_1 : (u_1, v_1, t_1) \to (u_1^2, v_1, u_1 v_1, t_1) = (x, y, z, w),$$
$$f_2 : (u_2, v_2, t_2) \to (u_2, v_2, t_2, 0) = (x, y, z, w).$$

Here (u_i, v_i, t_i) $(i = 1, 2)$ is a complex analytic local coordinate with center q_i. Then the pull-backs of the defining equation of H_λ in (12) by f_i $(i = 1, 2)$ are given by

$$A_1^* u_1^2 + B_1^* v_1 + C_1^* u_1 v_1 + D_1^* t_1 = 0, \quad \text{and} \tag{13}$$

$$A_2^* u_1 + B_2^* v_2 + C_2^* t_2 = 0 \tag{14}$$

where A_i^*, B_i^*, C_i^* and D_i^* $(i = 1, 2)$ are the pull-backs of A, B, C and D by the map f_i. The equations above give the defining equations of $\widetilde{X_\lambda}$ at q_1 and q_2, respectively. Concerning the equation (13), if $B_1^*(0) \neq 0$ or $D_1^*(0) \neq 0$, then $\widetilde{X_\lambda}$ is non-singular at q_1. Assume $B_1^*(0) = D_1^*(0) = 0$ to the contrary, then $B(0) = D(0) = 0$. Since

$$A(0)x + B(0)y + C(0)z + D(0)w = 0$$

is the equation of the embedded tangent space to H_λ at p in terms of the local coordinate (x, y, z, w), and since H_λ is defined by the equation (4), we have

$$\sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) = B(0) = 0, \ \text{ and } \ \sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) = D(0) = 0.$$

On the other hand, since $\lambda \notin B$, this is because of the condition (10), we have

$$\sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) \neq 0, \ \text{ or } \ \sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) \neq 0.$$

This is a contradiction. Therefore we conclude that $B_1^*(0) \neq 0$ or $D_1^*(0) \neq 0$, and so $\widetilde{X_\lambda}$ is non-singular at q_1. Concerning the equation (14), if $A_2^*(0) = B_2^*(0) = C_2^*(0) = 0$, then $A(0) = B(0) = C(0) = 0$. This means the equation of the embedded tangent space to H_λ at p with respect to the local coordinate (x, y, z, w) is $w = 0$, that is, H_λ is tangent to the hypersurface $w = 0$ at p. But this is a contradiction, because, since $\lambda \notin X^\vee$, H_λ is not a limit of tangent hyperplanes to X in $P^4(\mathbf{C})$ at simple points of X. Therefore we conclude that at least one of $A_2^*(0)$, $B_2^*(0)$ and $C_2^*(0)$ is not zero, and so $\widetilde{X_\lambda}$ is non-singular at q_2.

The assertion (ii): From the same reasoning as in the proof of the assertion (i) it follows that $\widetilde{X_{\lambda_i}}$ is non-singular at every point of $f^{-1}(D_{\lambda_i})$ where $D_{\lambda_i} = X_{\lambda_i} \cap D$. Hence it suffices to show that X_{λ_i} has only one isolated ordinary double point on $X_{\lambda_i} \cap X_{sm}$. By the manner of choosing the line L^* in $P^4(\mathbf{C})^\vee$, the hyperplane H_{λ_i} is tangent to X at only one point, say q, of X_{sm}. Therefore X_{λ_i} is non-singular at all but one point q of $X_{\lambda_i} \cap X_{sm}$. To prove that X_{λ_i} has an isolated ordinary double point at q, we assume that the homogeneous coordinate $[x_0 : x_1 : x_2 : x_3 : x_4]$ of q is $[1 : 0 : 0 : 0 : 0]$ and H_{λ_i} is defined by $x_4 = 0$. We put $\zeta_i = x_i/x_0$ $(1 \leq i \leq 4)$, and use this linear affine coordinate $(\zeta_1, \cdots, \zeta_4)$ in the subsequent arguments. Then X is defined by $F(1, \zeta_1, \zeta_2, \zeta_3, \zeta_4) = 0$, q is the origin $(0, \cdots, 0)$, and H_{λ_i} is defined by $\zeta_4 = 0$. Since the tangent hyperplane to X at q is the hyperplane $H_{\lambda_i} : \zeta_4 = 0$, we have

$$\frac{\partial F}{\partial \zeta_i}(1, 0, \cdots, 0) = 0 \qquad (1 \leq i \leq 3) \tag{15}$$

$$\frac{\partial F}{\partial \zeta_4}(1, 0, \cdots, 0) \neq 0. \tag{16}$$

Because of (16), there is an analytic function $\phi(\zeta_1, \zeta_2, \zeta_3)$ of the variables $\zeta_1, \zeta_2, \zeta_3$ defined in a neighborhood of the origin, which satisfies the following:

$$\phi(0,0,0) = 0, \tag{17}$$

$$F(1,\, \zeta_1,\, \zeta_2,\, \zeta_3,\, \phi(\zeta_1, \zeta_2, \zeta_3)) \equiv 0 \quad \text{(locally)}. \tag{18}$$

This means that the defining equation of X in a neighborhood of q is given by

$$\zeta_4 = \phi(\zeta_1, \zeta_2, \zeta_3) \tag{19}$$

By the same reasoning as before, we have

$$\frac{\partial \phi}{\partial \zeta_i}(0,0,0) = 0 \qquad (1 \leq i \leq 3) \tag{20}$$

Hence ϕ is expresed as

$$\phi = \sum_{1 \leq i,\, j \leq 3} \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j}(0)\, \zeta_i \zeta_j + O(3) \tag{21}$$

If we regard $(\zeta_1, \zeta_2, \zeta_3)$ as a local coordinate on H_{λ_i}, X_{λ_i} is defined by $\phi(\zeta_1, \zeta_2, \zeta_3) = 0$ in H_{λ_i}. Therefore, if we prove

$$\det\Big(\frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j}(0)\Big) \neq 0 \tag{22}$$

then we can conclude that, after suitable change of local coordinates, the defining equation of X_{λ_i} will become

$$u(\zeta_1, \zeta_2, \zeta_3)(\zeta_1^2 + \zeta_2^2 + \zeta_3^2) = 0$$

in a neighborhood of the origin in H_{λ_i}, where $u(\zeta_1, \zeta_2, \zeta_3)$ is a non-vanishing analytic function. This proves the assertion (ii) holds. To prove (22), we evaluate the Hessian $\det(\partial^2 F/\partial x_i \partial x_j)$ of the homogeneous polynomial F at $q = [1:0:0:0:0]$.

First we mention some remarks about $\det(\partial^2 F/\partial x_j \partial x_j(1,0))$, where and in what follows we write $(1,0)$ in stead of $(1,0,0,0,0)$ for short. From the Euler identity

$$\sum_{i=0}^{4} x_i \frac{\partial F}{\partial x_i} = n\, F \qquad (n = \deg F), \tag{23}$$

it follows that

$$\sum_{j=0}^{4} x_j \frac{\partial^2 F}{\partial x_i \partial x_j} = (n-1) \frac{\partial F}{\partial x_i} \qquad (0 \leq i \leq 4). \tag{24}$$

If $x_0 \neq 0$, by use of (24) and (23), we can derive

$$\det\Big(\frac{\partial^2 F}{\partial x_i x_j}\Big) = \Big(\frac{n-1}{x_0}\Big)^2 \begin{vmatrix} \frac{n}{n\text{-}1} F & \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_4} \\ \frac{\partial F}{\partial x_1} & \frac{\partial^2 F}{\partial x_1^2} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_4} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial F}{\partial x_4} & \frac{\partial^2 F}{\partial x_4 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_4^2} \end{vmatrix}. \tag{25}$$

Therfore, since $F(1,0)=0$ and $(\partial F/\partial x_i)(1,0)=0$ $(1\le i\le 3)$ (cf. (15)), we have

$$\det\Big(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\Big) = (n-1)^2\Big(\frac{\partial F}{\partial x_4}(1,0)\Big)^2 \det\Big(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\Big)_{1\le i,j\le 3} \tag{26}$$

Here we need to recall that $\Phi(q)=\lambda$ does not belong to $(X^\vee)_{sing}$ because of the condition (9). This means the Gauss map Φ defined by (1) gives a biregular morphism between X and $X^\vee$ in a neighborhood of q. Therefore the right-hand-side of (26) is not zero, and so we have

$$\det\Big(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\Big)_{1\le i,j\le 3} \neq 0 \tag{27}$$

since $(\partial F/\partial x_4)(1,0)\neq 0$ (cf. (16)). On the other hand, derivating the equation (18) twice with respect to the variables ζ_1,ζ_2,ζ_3 and substituting 0 for all ζ_i, we have

$$\det\Big(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\Big)_{1\le i,j\le 3} = -\Big(\frac{\partial F}{\partial x_4}(1,0)\Big)^3 \det\Big(\frac{\partial^2 \phi}{\partial \zeta_i \zeta_j}(1,0)\Big) \tag{28}$$

Since $(\partial F/\partial x_4)(1,0)\neq 0$, by (28) and (27) we have

$$\det\Big(\frac{\partial^2 \phi}{\partial \zeta_i \zeta_j}(1,0)\Big) \neq 0$$

as required. This completes the proof of the theorem. ∎

In what follows we assume that P_∞ is sufficiently general so that Theorem 2.1 holds.

LEMMA 2.2 *With the notation in Proposition 2.1, we have the following:*

(i) $\widetilde{C_\infty} := f^{-1}(C_\infty)$ *is a non-singular curve,*
(ii) $\widetilde{\mathcal{L}} := \bigcup_{\lambda\in P}\widetilde{X_\lambda}$ *is a linear system on* X_1 *with the base point locus* $B_s(\widetilde{\mathcal{L}})=\widetilde{C_\infty}$*, and*
(iii) *for* $\lambda,\mu\in P$ *with* $\lambda\neq\mu$*,* $\widetilde{X_\lambda}$ *and* $\widetilde{X_\mu}$ *intersect transversely along* $\widetilde{C_\infty}$*.*

PROOF. We take an affine coordinate neighborhood U of $P^4(\mathbf{C})$ with $U\cap P_\infty\neq\emptyset$, and work on this neighborhood. Let $(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$ be a linear affine coordinate on U. We may assume that

(29)
(a) $P_\infty\cap T=\emptyset$ and $P_\infty\cap C=\emptyset$,
(b) P_∞ and X intersect transversely at every non-singular point of X, and
(c) P_∞ and D intersect transversely.

Let $P_\infty = H_0\cap H_1$ where H_0 and H_1 are hyperplanes in $P^4(\mathbf{C})$, and let φ_i be a linear function which defines H_i on U for $i=1,2$. Note that the linear system $\widetilde{\mathcal{L}} := \bigcup_{\lambda\in P}\widetilde{X_\lambda}$ is defined by $\alpha f^*\varphi_0+\beta f^*\varphi_1$ $(\alpha,\beta\in\mathbf{C})$ where $f^*\varphi_i$ $(i=1,2)$ denotes the pull-back of φ_i by the normalization map $f: X_1\to X$. Therefore the assertion (ii) is trivial. By the assumption (29), (b), the assertions (i) and (iii) also trivially hold at $q=f^{-1}(p)$ for a non-singular point p of X, so we will prove that the assertions (i) and (iii) hold at $q\in f^{-1}(p)$ for $p\in D\cap U$. We assume that $\overline{X}$ is defined by $XY=0$ with respect to some complex analytic local coordinate (X,Y,Z,W) with center p, and assume that the normaliztion map f is given by

$$(u,v,t)\to(0,u,v,t)=(X,Y,Z,W),$$

where (u, v, t) is a complex analytic local coordinate with center $q := f^{-1}(p)$. The Jacobian matrix of $f^*\varphi_0, f^*\varphi_1$ with respect to (u, v, t) at q is given as follows:

$$\begin{aligned} &\frac{\partial(f^*\varphi_0, f^*\varphi_1)}{\partial(u,v,t)}(q) \\ &= \begin{pmatrix} \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial Y}(p)\frac{\partial \varphi_0}{\partial \zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial Z}(p)\frac{\partial \varphi_0}{\partial \zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial W}(p)\frac{\partial \varphi_0}{\partial \zeta_i}(p) \\ \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial Y}(p)\frac{\partial \varphi_1}{\partial \zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial Z}(p)\frac{\partial \varphi_1}{\partial \zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial W}(p)\frac{\partial \varphi_1}{\partial \zeta_i}(p) \end{pmatrix} \end{aligned} \tag{30}$$

On the other hand, by the assumption (29), (c),

$$\begin{vmatrix} \frac{\partial \varphi_0}{\partial Z}(p) & \frac{\partial \varphi_0}{\partial W}(p) \\ \frac{\partial \varphi_1}{\partial Z}(p) & \frac{\partial \varphi_1}{\partial W}(p) \end{vmatrix} \neq 0.$$

Hence,

$$\begin{vmatrix} \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial Z}(p)\frac{\partial \varphi_0}{\partial \zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial W}(p)\frac{\partial \varphi_0}{\partial \zeta_i}(p) \\ \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial Z}(p)\frac{\partial \varphi_1}{\partial \zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial \zeta_i}{\partial W}(p)\frac{\partial \varphi_1}{\partial \zeta_i}(p) \end{vmatrix} \neq 0. \tag{31}$$

By (30) and (31), we conclude $\{\partial(f^*\varphi_0, f^*\varphi_1)/\partial(u, v, t)\}(p)$ has the maximal rank. From this it follows that $\widetilde{C_\infty}$ is non-singular at q. Furthermore, if $[\alpha : \beta] \neq [\alpha' : \beta']$ as a point of $\mathrm{P}^1(\mathbf{C})$, then $\alpha\beta' - \alpha'\beta \neq 0$, so

$$\frac{\partial(f^*\varphi_0, f^*\varphi_1)}{\partial(u,v,t)}(q) \quad \text{and} \quad \frac{\partial(\alpha f^*\varphi_0 + \beta f^*\varphi_1,\ \alpha' f^*\varphi_0 + \beta' f^*\varphi_1)}{\partial(u,v,t)}(q)$$

have the same rank. Hence $\{\partial(\alpha f^*\varphi_0 + \beta f^*\varphi_1,\ \alpha' f^*\varphi_0 + \beta' f^*\varphi_1)/\partial(u, v, t)\}(q)$ has also the maximal rank. This shows that the assertion (iii) holds at q as required. This completes the proof of the lemma. ∎

Let $\sigma : \widehat{X_1} \to X_1$ be the blowing-up along $\widetilde{C_\infty} := f^{-1}(C_\infty)$, and $\widehat{\mathcal{L}} := \bigcup_{\lambda \in P} \widehat{X_\lambda}$ the proper inverse of $\widetilde{\mathcal{L}} := \bigcup_{\lambda \in P} \widetilde{X_\lambda}$. Then $\widehat{\mathcal{L}}$ gives a fibering of $\widehat{X_1}$ over $P \simeq P^1(\mathbf{C})$, which we denote by $\pi : \widehat{X_1} \to P$. Let $S = \{\lambda_1, \cdots, \lambda_c\}$ and $\widehat{X_1}^* = \widehat{X_1} - \pi^{-1}(S)$. From the exact Borel-Moore homology sequence determined by the space, the closed subspace, and its complement, it follows that

$$\chi(\widehat{X_1}) = \chi(\widehat{X_1}^*) + \chi(\pi^{-1}(S)). \tag{32}$$

It is clear that

$$\chi(\pi^{-1}(S)) = \sum_{i=1}^{c} \chi(\widehat{X_{\lambda_i}}). \tag{33}$$

Since $\widehat{X_1}^* \to P - S$ is locally trivial (as a differential fiber space), it follows from the spectral sequence of Leray for this fiber space that

$$\chi(\widehat{X_1}^*) = \chi(\widehat{X_\lambda})\chi(P - S), \tag{34}$$

where $\widehat{X_\lambda}$ denote a generic fiber of $\widehat{X_1}^* \to P - S$. By the same reason as before, we have

$$\chi(P) = \chi(P - S) + c. \tag{35}$$

Comparing (32), (33), (34) and (35), we have

$$\begin{aligned} \chi(\widehat{X_1}) &= \chi(P^1(\mathbf{C}))\chi(\widehat{X_\lambda}) + \Sigma_{j=1}^{c}(\chi(\widehat{X_{\lambda_j}}) - \chi(\widehat{X_\lambda})) \\ &= 2\chi(\widehat{X_\lambda}) - c. \end{aligned}$$

The second equality above follows from the fact that a topological 2-cycle vanishes when $\lambda \to \lambda_j$ for $j = 1, \cdots, c$. We put $\widehat{E} := \sigma^{-1}(\widetilde{C_\infty})$. Then, since $\widehat{X_1} \setminus \widehat{E} \simeq X_1 \setminus \widetilde{C_\infty}$,

$$\begin{aligned} \chi(\widehat{X_1}) - \chi(X_1) &= \chi(\widehat{E}) - \chi(\widetilde{C_\infty}) \\ &= \chi(P^1(C))\chi(\widetilde{C_\infty}) - \chi(\widetilde{C_\infty}) \\ &= \chi(\widetilde{C_\infty}) \end{aligned}$$

Hence,

$$\chi(X_1) = \chi(\widehat{X_1}) - \chi(\widetilde{C_\infty}) = 2\chi(\widehat{X_\lambda}) - \chi(\widetilde{C_\infty}) - c = 2\chi(\widetilde{X_\lambda}) - \chi(\widetilde{C_\infty}) - c. \tag{36}$$

Since C_∞ is a curve whose degree is equal to n with m ordinary double points in $P_\infty \simeq P^2(\mathbf{C})$, the genus $g(\widetilde{C_\infty})$ is given by

$$g(\widetilde{C_\infty}) = \frac{1}{2}(n-1)(n-2) - m.$$

Hence,

$$\chi(\widetilde{C_\infty}) = 2 - 2g(\widetilde{C_\infty}) = 2 - (n-1)(n-2) + 2m. \tag{37}$$

Note that X_λ is a surface with ordinary singularities in $H_\lambda \simeq P^3(\mathbf{C})$ of degree n, whose numerical characteristics related to its singularitis are as follows:

$$\text{the degree of its double curve } D_\lambda = m$$

$$\#\{\text{triple points of } X_\lambda\} = t, \quad \#\{\text{cuspidal points of } X_\lambda\} = \gamma.$$

Therefore, by the classical formula,

$$\chi(\widetilde{X_\lambda}) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma \tag{38}$$

By (36), (37) and (38), we have the following:

PROPOSITION 2.3

$$\chi(X_1) = 2n(n^2 - 4n + 6) - 2(3n - 8)m + 6t - 4\gamma$$

$$- 2 + (n-1)(n-2) - 2m - c$$
$$= \quad n(2n^2 - 7n + 9) - 2(3n-7)m + 6t - 4\gamma - c$$

3. The computation of the class of an algebraic threefold with ordinary singularities in $P^4(\mathbf{C})$. Throughout this section we denote a rational equivalence class of an algebraic cycle, say α, by $[\alpha]$, and denote the intersection class of two algebraic cycle classes, say $[\alpha]$ and $[\beta]$, by $\alpha \cdot \beta$. We refer to the following theorem from [5].

Theorem 3.1 ([5], Theorem (2.3)) *Let X^n be a hypersurface of degree d in P^{n+1}. Then its k-th polar class is given by*

$$[M_k] = [(d-1)c_1(L)]^k \cap [X] - \Sigma_{i=0}^{k-1} \binom{k}{i} [(d-1)c_1(L)]^i \cap s_{n-k+i}(J, X) \qquad (0 \leq k \leq n)$$

where $L = \mathcal{O}_{P^n}(1)$ and $s(J, X) = \Sigma_{k=0}^{n} s_k(J, X)$ $(s_k(J, X) \in A_k(J))$ denotes the Segre class of the singular subscheme J of X.

In what follows, using the theorem above, we shall compute the *class* c of an algebraic threefold with ordinary singularities in the complex projective 4-space $P^4(\mathbf{C})$ for the case where the threefold is free from quadruple points. First we fix the notation as follows:

$Y = P^4(\mathbf{C})$ **:** the complex projective 4-space,
$\overline{X}$ **:** an algebraic threefold with ordinary singularities in Y, which is free from quadruple points,
$\overline{J}$ **:** the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$,
$\overline{D}$ **:** the singular locus of $\overline{X}$,
$\overline{T}$ **:** the triple point locus of $\overline{X}$, which is equal to the singular locus of $\overline{D}$,
$\overline{C}$ **:** the cuspidal point locus of $\overline{X}$, precisely, its closure, since we always consider $\overline{C}$ contains the stationary points,
$\Sigma\overline{s}$ **:** the stationary point locus of $\overline{X}$,
$n_{\overline{X}} : X \to \overline{X}$ **:** the normalization of $\overline{X}$,
$f : X \to Y$ **:** the composite of the normaliztion map $n_{\overline{X}}$ and the inclusion $\iota : \overline{X} \hookrightarrow Y$,
J **:** the scheme-theoretic inverse of $\overline{J}$ by f,
D**,** T**,** C and Σs **:** the inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma\overline{s}$ by f, respectively.

Note that $\overline{T}$ and $\overline{C}$ are non-singular curves, intersecting transversely at $\Sigma\overline{s}$, and that the normalization X of $\overline{X}$ is also non-singular. Calculating by use of local coordinates, we can easily see the following:

(i) J contains D, and the *residual scheme* (cf. [2], Definition 9.2.1, p. 160) to D in J is the reduced scheme C;
(ii) D is a surface with ordinary singularities, free from triple points, whose singular locus is T,
(iii) D is the *double point* locus of the map $f : X \to Y$, i.e., the closure of $\{q \in X \mid \# f^{-1}(f(q)) \geq 2\}$;
(iv) the map $f_{|D} : D \to \overline{D}$ is generically two to one, simply ramified at C;
(v) the map $f_{|T} : T \to \overline{T}$ is generically three to one, simply ramified at Σs.

To compute the Segre class $s(J, X)$, the following proposition is useful.

Proposition 3.2 ([2], Proposition 9.2, p. 161) *Let $D \subset W \subset V$ be closed embeddings of schemes, with V a k-dimensional variety, and D a Cartier divisor on V. Let R be the residual scheme to D in W. Then, for all m,*

$$s(W,V)_m = s(D,V)_m + \Sigma_{j=0}^{k-m} \binom{k-m}{j} [-D]^j \cdot s(R,V)_{m+j}$$

in $A_m(W)$, the m-th ratinal equivalence class group of algebraic cycles on W.

In our case, since $D = f^{-1}(\overline{D})$ is a Cartier divisor, its normal cone $C_D X$ to D in X is isomorphic to $\mathcal{O}_X(D)_{|D}$, the restriction to D of the line bundle $\mathcal{O}_X(D)$ associated to D. Therefore,

$$\begin{aligned} s(D,X) &= c(\mathcal{O}_X(D)_{|D})^{-1} \cap [D] \\ &= [D] - c_1(\mathcal{O}_X(D)_{|D}) \cap [D] + c_1(\mathcal{O}_X(D)_{|D})^2 \cap [D] \\ &= [D] - [D]^2 + [D]^3. \end{aligned}$$

Since C is non-singular,

$$c(N_{C/X})^{-1} \cap [C] = [C] - c_1(N_{C/X}) \cap [C].$$

Hence, applying Proposition 3.2 for $W = J$, $D = f^{-1}(\overline{D})$ and $R = C$, we have

$$\left\{ \begin{array}{l} s(J,X)_2 = [D] \\ s(J,X)_1 = -[D]^2 + [C] \\ s(J,X)_0 = [D]^3 - c_1(N_{C/X}) \cap [C] - 3D \cdot C \end{array} \right. \tag{39}$$

Since $s(\overline{J},\overline{X})_2 = f_* s(J,X)_2$, from the first equality above, it follows that

$$s(\overline{J},\overline{X})_2 = 2[\overline{D}] \tag{40}$$

To know $s(\overline{J},\overline{X})_1$, we need to understand $f_*[D]^2$, the push-forward of $[D]^2$ by f. For this purpose, we compute $f^*[D]^2$. To compute this, we concider the following fiber square:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle \tau_T}\downarrow & & \downarrow{\scriptstyle \sigma_{\overline{T}}} \\ X & \xrightarrow[f]{} & Y. \end{array} \tag{41}$$

Here

$\sigma_{\overline{T}} : Y' \to Y$ **:** the blowing-up of Y along the triple point locus $\overline{T}$ of $\overline{X}$,
$\overline{X'}$ **:** the proper inverse image of $\overline{X}$ by $\sigma_{\overline{T}}$,
$X' := X \times_{\overline{X}} \overline{X'}$ **:** the fiber product of X and $\overline{X'}$ over $\overline{X}$,
$n_{\overline{X'}} : X' \to \overline{X'}$ **:** the projection to the second factor of $X \times_{\overline{X}} \overline{X'}$, which is nothing but the normalization of $\overline{X'}$,
$f' : X' \to Y'$ **:** the composite of the normaliztion map $n_{\overline{X'}}$ and the inclusion $\iota' : \overline{X'} \hookrightarrow Y'$,
$\tau_T : X' \to X$ **:** the projection to the first factor of $X \times_{\overline{X}} \overline{X'}$, which is nothing but the blowing-up of X along T.

In what follows, we denote by $\overline{D'}$, $\overline{T'}$ and $\overline{C'}$ the proper inverse images of $\overline{D}$, $\overline{T}$ and $\overline{C}$ by $\sigma_{\overline{T}}$, respectively. We consider the following fiber square:

$$\begin{array}{ccc} E_{\overline{T}} & \xrightarrow{\overline{j}} & Y' \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle \sigma_{\overline{T}}} \\ \overline{T} & \xrightarrow[\iota]{} & Y, \end{array} \tag{42}$$

where $E_{\overline{T}} = P(N_{\overline{T}}Y)$ is the exceptional divisor of the blowing-up $\sigma_{\overline{T}}$, which is a $P^2(\mathbf{C})$-bundle on $\overline{T}$, and $\overline{p} : E_{\overline{T}} \to \overline{T}$ is the projection to the base space of this bundle. We denote by $\mathcal{O}_{N_{\overline{T}}Y}(1)$ the canonical line bundle on $E_{\overline{T}}$. Then the *tautological line bundle* on $E_{\overline{T}}$ is $\mathcal{O}_{N_{\overline{T}}Y}(-1)$, which is a subbundle of $\overline{p}^* N_{\overline{T}}Y$.

LEMMA 3.3 *$\sigma_{\overline{T}}^*[\overline{D}]$ is expressed as*

$$\sigma_{\overline{T}}^*[\overline{D}] = [\overline{D'}] + 3\overline{j}_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*[\alpha_0] \tag{43}$$

where $[\xi_{\overline{T}}] = c_1(\mathcal{O}_{N_{\overline{T}}Y}(1)) \cap [E_{\overline{T}}]$ and $[\alpha_0]$ an algebraic 0-cycle class on $\overline{T}$.

PROOF. By the *blow-up formula*([2], Theorem 6.7, p.116),

$$\sigma_{\overline{T}}^*[\overline{D}] = [\overline{D'}] + \overline{j}_*\{c(E) \cap \overline{p}^* s(\overline{T}, \overline{D})\}_2 \tag{44}$$

where $E = \overline{p}^* N_{\overline{T}}Y / N_{E_{\overline{T}}}Y' = \overline{p}^* N_{\overline{T}}Y / \mathcal{O}_{N_{\overline{T}}Y}(-1)$. Since

$$c_1(E) = \overline{p}^* c_1(N_{\overline{T}}Y) - c_1(\mathcal{O}_{N_{\overline{T}}Y}(-1)) = \overline{p}^* c_1(N_{\overline{T}}Y) + c_1(\mathcal{O}_{N_{\overline{T}}Y}(1)),$$

we have

$$\begin{aligned} \{c(E) \cap s(\overline{T}, \overline{D})\}_2 &= \overline{p}^* s_0(\overline{T}, \overline{D}) + c_1(E) \cap \overline{p}^* s_1(\overline{T}, \overline{D}) \\ &= \overline{p}^*\{s_0(\overline{T}, \overline{D}) + c_1(N_{\overline{T}}Y) \cap s_1(\overline{T}, \overline{D})\} \\ &\qquad + c_1(\mathcal{O}_{N_{\overline{T}}Y}(1)) \cap \overline{p}^* s_1(\overline{T}, \overline{D}) \end{aligned} \tag{45}$$

To compute $s(\overline{T}, \overline{D})$, we consider the normalization map $n_{\overline{D}} : \overline{D}^* \to \overline{D}$. $\overline{D}^*$ is non-singular. Hence, if we put $\overline{T}^* := n_{\overline{D}}^{-1}(\overline{T})$, we have

$$\begin{aligned} s(\overline{T}^*, \overline{D}^*) &= c(N_{\overline{T}^*}\overline{D}^*)^{-1} \cap [\overline{T}^*] \\ &= (1 - c_1(N_{\overline{T}^*}\overline{D}^*)) \cap [\overline{T}^*] \\ &= [\overline{T}^*] - \overline{T}^* \cdot \overline{T}^*. \end{aligned}$$

Therefore,

$$s(\overline{T}, \overline{D}) = n_{\overline{D}_*} s(\overline{T}^*, \overline{D}^*) = 3[\overline{T}] - n_{\overline{D}_*}(\overline{T}^* \cdot \overline{T}^*),$$

and so,

$$\begin{cases} s_0(\overline{T}, \overline{D}) = -n_{\overline{D}_*}(\overline{T}^* \cdot \overline{T}^*) \\ s_1(\overline{T}, \overline{D}) = 3[\overline{T}] \end{cases} \tag{46}$$

By (45) and (46), if we put $[\alpha_0] := -n_{\overline{D}_*}(\overline{T}^* \cdot \overline{T}^*) + 3c_1(N_{\overline{T}}Y) \cap [\overline{T}]$,

$$\{c(E) \cap s(\overline{T}, \overline{D})\}_2 = \overline{p}^*[\alpha_0] + 3[\xi_{\overline{T}}].$$

Cosequently, by (44), we have the equality in (43). ∎

PROPOSITION 3.4

$$[D]^2 = f^*[\overline{X}] \cdot D - f^*[\overline{D}] + [T] - [C] \tag{47}$$

Proof. To know $[D]^2$, we compute $f^*[\overline{D}]$. For this purpose, we use the diagram in (41). Since $\tau_T : X' \to X$ is a blowing-up, we have $\tau_{T_*}\tau_T^*\alpha = \alpha$ for any algebraic cycle $\alpha \in A_*(X)$. Hence,

$$\tau_{T_*} f'^* \sigma_{\overline{T}}^*[\overline{D}] = \tau_{T_*}\tau_T^* f^*[\overline{D}] = f^*[\overline{D}]. \tag{48}$$

Since $\overline{D'}$ is regularly embedded in Y', i.e., $C_{\overline{D'}}Y' \simeq N_{\overline{D'}}Y'$, while $\overline{D}$ is not, we can apply the *excess intersection formula* ([2, Theorem 6.3, p.102]) to $\overline{D'}$. Then, denoting the tangent bundle of a non-singular algebraic variety, say Z, by $\mathcal{T}_Z$ we have

$$\begin{aligned} f'^*[\overline{D'}] &= c_1(f'^* N_{\overline{D'}}Y'/N_{D'}X') \cap [D'] \\ &= \{c_1(f'^*\mathcal{T}_{Y'}) - c_1(f'^*\mathcal{T}_{\overline{D'}}) - c_1(\mathcal{T}_{X'}) + c_1(\mathcal{T}_{D'})\} \cap [D'] \\ &= \{c_1(f'^*\mathcal{T}_{Y'}) - c_1(\mathcal{T}_{X'})\} \cap [D'] - C', \end{aligned} \tag{49}$$

where the last equality follows from the *ramification formula* ([2, Example 3.2.20, p.62]). On the other hand, by the *double point formula* ([2, Theorem 9.3, p.166]),

$$[D'] = f'^*[\overline{X'}] - \{c_1(f'^*\mathcal{T}_{Y'}) - c_1(\mathcal{T}_{X'})\} \cap [X']. \tag{50}$$

By (49) and (50), we have

$$f'^*[\overline{D'}] = f'^*[\overline{X'}] \cdot D' - [D']^2 - C'. \tag{51}$$

Next, in view of Lemma 3.3, we compute $f'^*(3\overline{j}_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*[\alpha_0])$. For this purpose, we consider the following fiber square:

$$\begin{array}{ccc} E_T & \xrightarrow{\ j\ } & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tau_T} \\ T & \xrightarrow[\ \iota\ ]{} & X, \end{array} \tag{52}$$

where $E_T = P(N_T X)$ is the exceptional divisor of the blowing-up τ_T, which is a $P^1(C)$-bundle on T, and $p : E_T \to T$ is the projection to the base space of this bundle. There is a set of morphisms from the diagram in (52) to the one in (42) induced by those in the diagram in (41). We denote by g and g' the restriction of $f : X \to Y$ to T and that of $f' : X' \to Y'$ to E_T, respectively. Note that the morphism $g' : E_T \to E_{\overline{T}}$ maps each fiber of $p : E_T \to T$ to that of $\overline{p} : E_{\overline{T}} \to \overline{T}$, and so $g'^*[\xi_{\overline{T}}] = [\xi_T]$ where $\xi_T = c_1(\mathcal{O}_{N_T X}(1)) \cap [E_T]$. Since $f' : X' \to Y'$ and $g' : E_T \to E_{\overline{T}}$ are local complete intersection morphisms of the same codimension, we can apply Proposition 6.6, (c) in [2, p.113] to the fiber square

$$\begin{array}{ccc} E_T & \xrightarrow{\ g'\ } & E_{\overline{T}} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \overline{j}} \\ X' & \xrightarrow[\ f'\ ]{} & Y'. \end{array} \tag{53}$$

Then, $f'^*\overline{j}_*[\xi_{\overline{T}}] = j_*g'^*[\xi_{\overline{T}}] = j_*[\xi_T]$ and $f'^*\overline{j}_*\overline{p}^*[\alpha_0] = j_*g'^*\overline{p}^*[\alpha_0] = j_*p^*g^*[\alpha_0]$. Therefore, we have

$$f'^*(3\overline{j}_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*[\alpha_0]) = 3j_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*g^*[\alpha_0] \tag{54}$$

By (43), (51) and (54), we have

$$f'^*\sigma_{\overline{T}}^*[\overline{D}] = f'^*[\overline{X'}] \cdot D' - [D']^2 - C' + 3j_*[\xi_{\overline{T}}] + j_*\overline{p}^*g^*[\alpha_0].$$

Since $\tau_{T_*}[C'] = [C]$, $\tau_{T_*} j_*[\xi_{\overline{T}}] = T$ and $\tau_{T_*} j_* \overline{p}^* g^*[\alpha_0] = 0$, by the equality above and (48),

$$\begin{aligned} f^*[\overline{D}] &= \tau_{T_*} f'^* \sigma^*_{\overline{T}}[\overline{D}] \\ &= \tau_{T_*}(f'^*[\overline{X'}] \cdot D') - \tau_{T_*}[D']^2 - [C] + 3[T]. \end{aligned} \tag{55}$$

Since $\tau_T^*[D] = [D'] + 2[E_T]$,

$$\tau_{T_*}(f'^*[\overline{X'}] \cdot D') = \tau_{T_*}(f'^*[\overline{X'}] \cdot \tau_T^*[D] - 2f'^*[\overline{X'}] \cdot E_T) \tag{56}$$

On the other hand, since $\sigma^*_{\overline{T}}[\overline{X}] = [\overline{X'}] + 3[E_{\overline{T}}]$,

$$f'^*[\overline{X'}] = f'^* \sigma_T^*[\overline{X}] - 3[E_T].$$

Hence, by the *projection formula*,

$$\begin{aligned} \tau_{T_*}(f'^*[\overline{X'}] \cdot \tau_T^*[D]) &= \tau_{T_*}(f'^*[\overline{X'}]) \cdot D \\ &= \tau_{T_*}(f'^* \sigma^*_{\overline{T}}[\overline{X'}]) \cdot D \\ &= f^*[\overline{X}] \cdot D, \end{aligned} \tag{57}$$

and

$$\begin{aligned} \tau_{T_*}(f'^*[\overline{X'}] \cdot E_T) &= \tau_{T_*}(f'^* \sigma^*_{\overline{T}}[\overline{X}] \cdot E_T - 3[E_T]^2) \\ &= \tau_{T_*}(\tau_T^* f^*[\overline{X}] \cdot E_T) + 3\tau_{T_*} j_*[\xi_T] \\ &= f^*[\overline{X}] \cdot \tau_{T_*}[E_T] + 3i_*[T] = 3[T] \end{aligned} \tag{58}$$

Therefore, by (56), (57) and (58),

$$\tau_{T_*}(f'^*[\overline{X'}] \cdot D') = f^*[\overline{X}] \cdot D - 6[T]. \tag{59}$$

Furthermore, we have

$$\begin{aligned} \tau_{T_*}[D']^2 &= \tau_{T_*}((\tau_T^*[D] - 2[E_T])^2) \\ &= \tau_{T_*}((\tau_T^*[D])^2 - 4\tau_T^*[D] \cdot [E_T] + 4[E_T]^2) \\ &= \tau_{T_*}(\tau_T^*[D]) \cdot D - 4D \cdot \tau_{T_*}[E_T] - 4\tau_{T_*} j_*[\xi_T] \\ &= [D]^2 - 4[T]. \end{aligned} \tag{60}$$

Consequently, by (55), (59) and (60),

$$\begin{aligned} f^*[\overline{D}] &= f^*[\overline{X}] \cdot D - 6[T] - [D]^2 + 4[T] - [C] + 3[T] \\ &= f^*[\overline{X}] \cdot D - [D]^2 - [C] + [T], \end{aligned}$$

from which the equality (47) follows. ∎

Since $f_*[X] = [\overline{X}]$, $f_*[D] = 2[\overline{D}]$, $f_*[T] = 3[\overline{T}]$ and $f_*[C] = [\overline{C}]$, by Proposition 3.4, we have the following:

Corollary 3.5

$$f_*[D]^2 = \overline{X} \cdot \overline{D} + 3[\overline{T}] - [\overline{C}] \tag{61}$$

By Proposition 3.4 and the second equality in (39),

$$s(J, X)_1 = -f^*[\overline{X}] \cdot D + f^*[\overline{D}] - [T] + 2[C]$$

and so, by the *projection formula*

$$s(\overline{J},\overline{X})_1 = -\overline{X}\cdot\overline{D} - 3[\overline{T}] + 2[\overline{C}] \tag{62}$$

Now we compute $s(\overline{J},\overline{X})_0$. By Proposition 3.4,

$$[D]^3 = f^*[\overline{X}]\cdot[D]^2 - f^*[\overline{D}]\cdot D + D\cdot T - D\cdot C$$

Hence, by the third equality in (39),

$$s(J,X)_0 = f^*[\overline{X}]\cdot[D]^2 - f^*[\overline{D}]\cdot D + D\cdot T - 4D\cdot C - c_1(N_CX)\cap[C] \tag{63}$$

Since $\overline{T}$ and $\overline{C}$ are regulary embedded in Y, we can apply the *excess intersection formula* to them. Then,

$$\begin{aligned} f^*[\overline{T}] &= c_1(f^*N_{\overline{T}}Y/N_TX)\cap[T] \\ &= \{c_1(f^*\mathcal{T}_Y) - c_1(f^*\mathcal{T}_{\overline{T}}) - c_1(\mathcal{T}_X) + c_1(\mathcal{T}_T)\}\cap[T] \\ &= \{c_1(f^*\mathcal{T}_Y) - c_1(\mathcal{T}_X)\}\cap[T] - [\Sigma s] \\ &= f^*[\overline{X}]\cdot T - D\cdot T - [\Sigma s], \end{aligned}$$

where the last step but one follows from the *ramification formula* for $g : T \to \overline{T}$ and the last step from the *double point formula* for $f : X \to Y$. Similarly, since $\overline{C}\simeq C$, we have

$$f^*[\overline{C}] = f^*[\overline{X}]\cdot C - D\cdot C$$

Therefore we have

$$\begin{cases} D\cdot T = f^*[\overline{X}]\cdot T - f^*[\overline{T}] - [\Sigma s] \\ D\cdot C = f^*[\overline{X}]\cdot C - f^*[\overline{C}] \end{cases} \tag{64}$$

By the *adjunction formula*, the *double point formula* for $f : X \to Y$ and the second equality in (64),

$$\begin{aligned} c_1(N_CX)\cap[C] &= -K_X\cdot C + [k_C] \\ &= (-f^*[\overline{X} + K_Y] + D)\cdot C + [k_C] \\ &= -f^*[K_Y]\cdot C - f^*[\overline{C}] + [k_C], \end{aligned} \tag{65}$$

where K_Y, K_X and k_C are the canonical divisors of Y, X and C, respectively. Substituting (64) and (65) into (63), we have

$$\begin{aligned} s(J,X)_0 \;=\; & f^*[\overline{X}]\cdot[D]^2 - f^*[\overline{D}]\cdot D + f^*[\overline{X}]\cdot T - f^*[\overline{T}] - [\Sigma s] \\ & - 4f^*[\overline{X}]\cdot C + 4f^*[\overline{C}] + f^*[K_Y]\cdot C + f^*[\overline{C}] - [k_C]. \end{aligned}$$

Consequently, using Corollary 3.5 and the fact that $f_*[X] = [\overline{X}]$, $f_*[D] = 2[\overline{D}]$, $f_*[T] = 3[\overline{T}]$, $f_*[\Sigma s] = [\Sigma\overline{s}]$ and $\overline{C}\simeq C$, we have,

$$s(\overline{J},\overline{X})_0 = [\overline{X}]^2\cdot\overline{D} - 2[\overline{D}]^2 + 5\overline{X}\cdot\overline{T} + K_Y\cdot\overline{C} - [k_{\overline{C}}] - [\Sigma\overline{s}].$$

We collect the results obtained till now in the following proposition:

PROPOSITION 3.6 *The Segre classes of the singular subscheme $\overline{J}$, defined by the Jacobian ideal, of an algebraic threefold $\overline{X}$ with ordinary singularities in the four dimensional* *projective space $Y = P^4(C)$ are given as follows, if $\overline{X}$ is free from quadruple points:*

$$\begin{cases} s(\overline{J},\overline{X})_2 = 2[\overline{D}] \\ s(\overline{J},\overline{X})_1 = -\overline{X}\cdot\overline{D} - 3\overline{T} + 2\overline{C} \\ s(\overline{J},\overline{X})_0 = [\overline{X}]^2\cdot\overline{D} - 2[\overline{D}]^2 + 5\overline{X}\cdot\overline{T} + K_Y\cdot\overline{C} - [k_{\overline{C}}] - [\Sigma\overline{s}] \end{cases}$$

Here $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma\overline{s}$ are the singular locus, triple point locus, cuspidal point locus and stationary point locus of $\overline{X}$, respectively. K_Y is the canonical divisor of the projective 4-space Y, and $k_{\overline{C}}$ that of $\overline{C}$.

4. The Euler number of the normalization of an algebraic threefold with ordinary singularities. By Theorem 3.1, the top polar calss $[M_3]$ of $\overline{X}$ is given by

$$[M_3] = (n-1)^3h^3 - 3(n-1)^2h^2\cap s_2 - 3(n-1)h\cap s_1 - s_0,$$

where h denotes the hyperplane section class and s_i i-th Segre class $s(\overline{J},\overline{X})_i$ $(0\leq i\leq 2)$ and $n = deg\,\overline{X}$, the degree of $\overline{X}$ in Y. We put

$$m = deg\,\overline{D},\ t = deg\,\overline{T},\ \gamma = deg\,\overline{C} \text{ and } \#\Sigma\overline{s} = \text{the cardinal number of } \Sigma\overline{s}$$

Then, by Proposition 3.6,

$$\begin{cases} deg\,s_2 = 2m \\ deg\,s_1 = -nm + 2\gamma - 3t \\ deg\,s_0 = n^2m - 2m^2 + 5nt - 5\gamma - \#\Sigma\overline{s} - deg\,k_{\overline{C}}. \end{cases}$$

Consequently, the *class* c of $\overline{X}$ is given by

$$\begin{aligned} c &= deg[M_3] = (n-1)^3deg\,\overline{X} - 3(n-1)^2deg\,s_2 - 3(n-1)deg\,s_1 - deg\,s_0 \\ &= (n-1)^3n - (4n^2-9n-2m+6)m + (4n-9)t - (6n-11)\gamma + \#\Sigma\overline{s} + deg\,k_{\overline{C}}. \end{aligned}$$

By this formula together with Proposition 2.3, we have the following:

THEOREM 4.1 *The Euler number $\chi(X)$ of the non-singular normalization X of an algebraic threefold $\overline{X}$ with ordinary singularities in $P^4(\mathbf{C})$ which is free from quadruple points is given by*

$$\begin{aligned} \chi(X) &= -n(n^3-5n^2+10n-10) + (4n^2-15n-2m+20)m - (4n-15)t \\ &\qquad + (6n-15)\gamma - \#\Sigma\overline{s} - deg\,k_{\overline{C}}. \end{aligned}$$

Here $n = deg\,\overline{X}$, $m = deg\,\overline{D}$, $t = deg\,\overline{T}$ and $\gamma = deg\,\overline{C}$ are the degrees of $\overline{X}$, the singular locus, the triple point locus and the cuspidal point locus, respectively. $\#\Sigma\overline{s}$ is the cardinal number of the stationary point locus $\Sigma\overline{s}$, and $deg\,k_{\overline{C}}$ the degree of the canonical divisor of the cuspidal point locus $\overline{C}$.

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