THE EULER NUMBER OF THE NORMALIZATION OF AN ALGEBRAIC THREEFOLD WITH ORDINARY SINGULARITIES

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Abstract. By a classical formula due to Enriques, the Euler number $\chi(X)$ of the non-singular normalization $X$ of an algebraic surface $S$ with ordinary singularities in $\mathbb{P}^3(\mathbb{C})$ is given by $\chi(X) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma$, where $n =$ the degree of $S$, $m =$ the degree of the double curve (singular locus) $D_S$ of $S$, $t =$ the cardinal number of the triple points of $S$, and $\gamma =$ the cardinal number of the cuspidal points of $S$. In this article we shall give a similar formula for an algebraic threefold with ordinary singularities in $\mathbb{P}^4(\mathbb{C})$ which is free from quadruple points (Theorem 4.1).

1. Preliminaries. We begin with recalling some definitions.

Definition 1. ([1]) An irreducible hypersurface $S$ in the complex projective 3-space $\mathbb{P}^3(\mathbb{C})$ is called an algebraic surface with ordinary singularities if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 3-space $\mathbb{C}^3$ at every point of $S$:

\[
\begin{align*}
(i) & \quad z = 0 \quad \text{(simple point)} \\
(ii) & \quad yz = 0 \quad \text{(ordinary double point)} \\
(iii) & \quad xyz = 0 \quad \text{(ordinary triple point)} \\
(iv) & \quad xy^2 - z^2 = 0 \quad \text{(cuspidal point)},
\end{align*}
\]

where $(x, y, z)$ is the coordinate on $\mathbb{C}^3$.

Definition 2. ([6]) An irreducible hypersurface $T$ in the complex projective 4-space $\mathbb{P}^4(\mathbb{C})$ is called an algebraic threefold with ordinary singularities if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbb{C}^4$ at

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[1]
We denote by \( I \) the graded Chow group of every point of \( T \):

\[
\begin{cases}
(i) \ w = 0 \ (\text{simple point}) & (ii) \ zw = 0 \ (\text{ordinary double point}) \\
(iii) \ yzw = 0 \ (\text{ordinary triple point}) & (iv) \ xyzw = 0 \ (\text{ordinary quadruple point}) \\
(v) \ xy^2 - z^2 = 0 \ (\text{cuspidal point}) & (vi) \ w(xy^2 - z^2) = 0 \ (\text{stationary point}),
\end{cases}
\]

where \((x, y, z, w)\) is the coordinate on \( \mathbb{C}^4 \).

It is known that every complex projective surface (resp. threefold) is birationally equivalent to an algebraic surface (resp. threefold) with ordinary singularities.

Next we give the definition of the \textit{polar classes} of an \( r \)-dimensional subvariety \( X^r \) in a complex projective space \( P^n(C) \). Denote by \( U \) the open subset of \( X^r \) consisting of all simple points of \( X \). For a given linear \((n-r+k-2)\) dimensional subspace \( L_{(k)} \) of \( P^n(C) \), we let \( M_k(U) \) denote the locus of points \( x \in U \) such that the tangent space \( T_x X \) at \( x \) intersects \( L_{(k)} \) in a space at least \( k-1 \) dimension.

**Definition 3.** The closure \( M_k \) of \( M_k(U) \) in \( X \) is called the \( k \)-th \textit{polar locus} of \( X \).

\( M_k \) has a natural reduced scheme structure and, for a general \( L_{(k)} \), \( M_k \) has codimension \( k \) in \( X \). Moreover, for such \( L_{(k)} \), the rational equivalent class of the cycle defined by \( M_k \) does not depend on \( L_{(k)} \) (cf. [5]).

**Definition 4.** This class denoted by \([M_k]\) is called the \( k \)-th \textit{polar class} of \( X \). The degree \( \mu_k \) of \( M_k \) is called the \( k \)-th \textit{class}. The top class \( \mu_r \) is called the \textit{class} of \( X \).

Now we give the definition of the \textit{Segre class} of a closed subscheme \( X \) of a scheme \( Y \). We denote by \( I \) the ideal sheaf of \( X \) in \( Y \) and put

\[
S := \Sigma_{k=0}^{\infty} I^k / I^{k+1},
\]

which is a graded sheaf of \( \mathcal{O}_X \)-algebras on \( X \). To \( S \) we associate two schemes over \( X \): the \textit{cone} of \( S \)

\[
C := \text{Spec}(S), \quad \pi : C \to X;
\]

and the \textit{projective cone} \( P(C) \) to \( X \) in \( Y \) by

\[
P(C) := \text{Proj}(S), \quad p : P(C) \to X.
\]

\( C \) is called the \textit{normal cone} to \( X \) in \( Y \), denoted by \( C_X Y \), and \( P(C) \) the \textit{projective normal cone} to \( X \) in \( Y \). On \( P(C) \) there is a canonical line bundle, denoted by \( \mathcal{O}_{C}(1) \). Let \( z \) be a variable, \( S[z] \) the graded algebra whose \( n \)-th graded piece \( (S[z])^n \) is

\[
S^n \oplus S^{n-1}z \oplus \cdots \oplus S^1 z^{n-1} \oplus S^0 z^n.
\]

The corresponding cone is denoted by \( C \oplus 1 \). The cone

\[
P(C \oplus 1) = \text{Proj}(S[z]), \quad q : P(C \oplus 1) \to X
\]

is called the \textit{projective completion} of \( C \). The element \( z \) in \( (S[z])^1 \) determines a regular section of \( \mathcal{O}_{C\oplus 1}(1) \) on \( P(C \oplus 1) \) whose zero-scheme is canonically isomorphic to \( P(C) \). The complement to \( P(C) \) in \( (C \oplus 1) \) is canonically isomorphic to \( C \).

**Definition 5.** The \textit{Segre class} of \( X \) in \( Y \), denoted by \( s(X, Y) \), is the class in the graded Chow group \( A_*(X) \) of \( X \) defined by the formula

\[
\text{Segre class of a closed subscheme } X \text{ of a scheme } Y.
\]
Let \( P \) is an isomorphism (cf. [2], Appendix B, B.7).

In the complex projective 4-space this section we denote by \( X \) locus of \( \Sigma \) is a locally free sheaf of rank \( d \in \lambda \) dimension \( d \) morphisms of schemes. \( \lambda \)

Note that \( s(X,Y) := q_*([\Sigma_{i \geq 0} c_i(\mathcal{O}_{C^{q+1}}(1)) \cap [P(C^{q+1})]]). \)

Finally, we give the definitions of \textit{regular embeddings} and \textit{local complete intersection morphisms} of schemes.

**Definition 6.** We say a closed embedding \( \iota : X \to Y \) of schemes is a \textit{regular embedding of codimension} \( d \) if every point in \( X \) has an affine neighborhood \( U \) in \( Y \), such that if \( A \) is the coordinate ring of \( U \), \( I \) the ideal of \( A \) defining \( X \), then \( I \) is generated by a regular sequence of length \( d \).

If this is the case, the conormal sheaf \( \mathcal{I}/\mathcal{T}^2 \), where \( \mathcal{I} \) is the ideal sheaf of \( X \) in \( Y \), is a locally free sheaf of rank \( d \). The \textit{normal bundle} to \( X \) in \( Y \), denoted by \( N_X/Y \), is the vector bundle on \( X \) whose sheaf of sections is dual to \( \mathcal{I}/\mathcal{T}^2 \). Note that the normal bundle \( N_X/Y \) is canonically isomorphic to the normal cone \( C_X \) \( Y \) for a (closed) regular embedding \( \iota : X \to Y \) since the canonical map from \( Sym(\mathcal{I}/\mathcal{T}^2) \to S := \Sigma_{k=1}^\infty \mathcal{T}^k/\mathcal{T}^{k+1} \) is an isomorphism (cf. [2], Appendix B, B.7).

**Definition 7.** A morphism \( f : X \to Y \) is called a \textit{local complete intersection morphism of codimension} \( d \) if \( f \) factors into a (closed) regular embedding \( \iota : X \to Y \) of some constant codimension \( e \), followed by a smooth morphism \( p : P \to Y \) of constant relative dimension \( d + e \).

2. The existence of a good linear pencil of hyperplane sections. Throughout this section we denote by \( X \) an algebraic threefold with ordinary singularities of degree \( n \) in the complex projective 4-space \( P^4(C) \), by \( D \) the double surface of \( X \), i.e., the singular locus of \( X \), by \( T \) the triple points locus of \( X \), by \( C \) the cuspidal point locus of \( X \), by \( \Sigma \) the stationary point locus of \( X \). Let \( m, t, \gamma \) be the degrees of \( D, T, C \), respectively. Let \( P_\infty \) be a 2-dimensional linear subspace of \( P^4(C) \) such that \( C_\infty := P_\infty \cap X \) is an irreducible curve with ordinary double points in \( P_\infty \simeq P^2(C) \). Let \( P \) be a 1-dimensional linear subspace of \( P^4(C) \) situated in twisted position with respect to \( P_\infty \), i.e., the linear subspace \( L(P_\infty, P) \) generated by \( P_\infty \) and \( P \) is equal to \( P^4(C) \). Let \( \pi : X \setminus C_\infty \to P \) be the linear projection with center \( C_\infty \), i.e., \( \pi(x) := H_x \cap P \) for \( x \in X \setminus C_\infty \), where \( H_x = L(x, P_\infty) \) is the hyperplane generated by \( x \) and \( P_\infty \). We put \( X_\lambda := H_x \cap X \) for \( \lambda \in P \) and \( L := \bigcup_{\lambda \in P} X_\lambda \). Then \( L \) is a linear system on \( X \) with the base point locus \( Bs(L) = C_\infty \). Let \( f : X_1 \to X \) be the normalization map and \( \tilde{L} := \bigcup_{\lambda \in P} \tilde{X}_\lambda \) the pull-back of \( L \) to \( X_1 \).

**Theorem 2.1** If we take \( P_\infty \) sufficiently general, there exists a finite set \( \{\lambda_1, \ldots, \lambda_c\} \) of points of \( P \) such that

(i) \( \tilde{X}_\lambda \) is non-singular for \( \lambda \) with \( \lambda \neq \lambda_i \) (1 \( \leq i \leq c \)), and
(ii) $\overline{X}_X$ is a surface with only one isolated ordinary double point which is contained in $X_1 \setminus f^{-1}(C_\infty)$ for any $i$ with $1 \leq i \leq c$.

where $c$ is the class of $X$.

PROOF. We consider the Gauss map

$$\Phi : X \to P^4(C)^\vee$$

defined by

$$\Phi(p) = \left[ \frac{\partial F}{\partial x_0}(p) : \frac{\partial F}{\partial x_1}(p) : \frac{\partial F}{\partial x_2}(p) : \frac{\partial F}{\partial x_3}(p) : \frac{\partial F}{\partial x_4}(p) \right]$$

for $p \in X$, where $F$ is the homogeneous polynomial defining $X$ in $P^4(C)$, $[x_0 : x_1 : x_2 : x_3 : x_4]$ the homogeneous coordinate on $P^4(C)$, and $P^4(C)^\vee$ the dual projective space of $P^4(C)$. $\Phi$ is a rational map, which is not defined on the singular locus $D$ of $X$. Let $\overline{X}$ be the closure in $X \times P^4(C)^\vee$ of the graph of $\Phi$. We denote by $\pi_1 : \overline{X} \to X$ the morphism induced by the projection to the first factor, and $\pi_2 : \overline{X} \to P^4(C)^\vee$ the one induced by the projection to the second factor. We call $\pi_1 : \overline{X} \to X$ the Nash blow-up of $X$. Note that the rational map $\Phi$ can be extended to $\overline{X}$ and $\overline{X}$ is minimal among the varieties with such property. In our case, since $X$ is a hypersurface, $\overline{X}$ coincides with the blow-up of the Jacobian ideal of $X$ ([4], Remark 2, p.300). We denote by $X^\vee$ the image of $\overline{X}$ by $\pi_2 : \overline{X} \to P^4(C)^\vee$, and call it the dual variety of $X$. The dimension of $X^\vee$ is not less than 1, nor greater than 3 ([3], Example 15.22., p.196).

We are now going to define an algebraic subset $B$ in $P^4(C)^\vee$, whose points correspond to hyperplanes in $P^4(C)$ being in bad positions in some sense at their intersecting points with the cuspidal point locus $C$, or stationary point locus $\Sigma s$ of $X$. Let $p$ be a point of $C$, or $\Sigma s$. Then there is an open neighborhood $U$ of $p$ and a complex analytic local coordinates $(x, y, z, w)$ with center $p$ such that the defining equation of $X$ is given by one of the following:

$$xy^2 - z^2 = 0$$

$$w(xy^2 - z^2) = 0.$$  

Let $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ be a linear affine coordinate with center $p$, and $H$ a hyperplane passing through $p$, defined by the equation

$$\sum_{i=1}^{4} a_i \zeta_i = 0 \quad (a_i \in \mathbb{C}, \ 1 \leq i \leq 4).$$

We say $H$ is in a bad position at the point $p$, if the coefficients of the equation (4) satisfy the following two conditions:

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) = 0,$$

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) = 0.$$
We define $B_p$ to be the algebraic subset of $P^4(C)\vee$ consisting of all points which corresponds to hyperplanes in $P^4(C)$ passing through $p$ and being in a bad position at $p$ in the sense defined above. We define an algebraic subset $B$ of $P^4(C)\vee$ by

$$B := \bigcup_{p \in C} B_p$$

Here we should note that the stationary points are included in $C$, and since dim$B_p = 1$, the codimension of $B$ is greater than 1. We choose a line $L^*$ in $P^4(C)\vee$ which satisfies all of the following conditions:

$$L^* \cap \{X^\vee \setminus \Phi(X_{sm})\} = \emptyset,$$

$$L^* \cap (X^\vee)_{sing} = \emptyset,$$

$$L^* \cap B = \emptyset,$$

$$L^* \text{ intersects transversely with } \Phi(X_{sm}) \setminus (X^\vee)_{sing},$$

where $X_{sm}$ denotes $X \setminus D$, the simple point locus of $X$, and $(X^\vee)_{sing}$ the singular point locus of $X^\vee$. This is always possible because all the codimensions of $X^\vee \setminus \Phi(X_{sm})$, $(X^\vee)_{sing}$ and $B$ are greater than 1 in $P^4(C)\vee$. Note that the cardinal number of the set $L^* \cap \{\Phi(X_{sm}) \setminus (X^\vee)_{sing}\}$ is nothing but the class of $X$. We denote by $H_\lambda$ the hyperplane in $P^4(C)$ corresponding to each $\lambda \in L^*$. We put $X_\lambda := X \cap H_\lambda$ and consider the linear pencil

$$\mathcal{L} = \bigcup_{\lambda \in L^*} X_\lambda$$

of hyperplane sections of $X$. We are now going to show that the assertions (i) and (ii) of the proposition hold for the pull-back $\tilde{\mathcal{L}} = \bigcup_{\lambda \in L^*} \tilde{X}_\lambda$ of $\mathcal{L}$ to the normal model $X_1$ of $X$ by the normalization map $f : X_1 \to X$.

The assertion (i): Let $\{\lambda_1, \ldots, \lambda_c\}$ be all of the distinct points of $L^* \cap \{\Phi(X_{sm}) \setminus (X^\vee)_{sing}\}$, and $\lambda$ a point $L^*$ with $\lambda \neq \lambda_i$ $(1 \leq i \leq c)$. Then $\lambda \notin X^\vee$. This means that $H_\lambda$ is not tangent to $X$ at any point of $X_{sm}$, and not a limit of tangent hyperplanes to $X_{sm}$. Hence we infer that $\tilde{X}_\lambda$ is non-singular at every point of $X_1 \setminus f^{-1}(C)$. Therefore what we have to do is to show that $\tilde{X}_\lambda$ is non-singular at $f^{-1}(p)$ for any point $p \in H_\lambda \cap C$. In the subsequence we shall show this fact only when $p$ is a statinary point, since the proof for a cuspidal point is more easy. Assume $p$ is a cuspidal point of $X$ and $p \in H_\lambda$. We take a complex analytic local coordinate $(x, y, x, w)$ with center $p$ such that the defining equation of $X$ is given by the equation (3). We also take a linear affine coordinate $(\zeta_0, \zeta_1, \zeta_2, \zeta_3)$ with center $p$ and assume that the defining equation of $H_\lambda$ is given by the equation (4). We rewrite the equation (4) as

$$Ax + By + Cz + Dw = 0,$$

where $A, B, C$ and $D$ are complex analytic functions defined in a neighborhood of $p$. $f^{-1}(p)$ is two points, say $q_1, q_2$, where the normalization map $f : X_1 \to X$ is given as follows:

$$f_1 : (u_1, v_1, t_1) \to (u_1^2, v_1, u_1v_1, t_1) = (x, y, z, w),$$

$$f_2 : (u_2, v_2, t_2) \to (u_2, v_2, t_2, 0) = (x, y, z, w).$$
Here \((u_i, v_i, t_i) \ (i = 1, 2)\) is a complex analytic local coordinate with center \(q_i\). Then the pull-backs of the defining equation of \(H_\lambda\) in (12) by \(f_i \ (i = 1, 2)\) are given by
\[
A_1^* u_1^2 + B_1^* v_1 + C_1^* u_1 v_1 + D_1^* t_1 = 0, \quad \text{and}
\]
\[
A_2^* u_1 + B_2^* v_2 + C_2^* t_2 = 0
\]
where \(A_i^*, B_i^*, C_i^*\) and \(D_i^* \ (i = 1, 2)\) are the pull-backs of \(A, B, C\) and \(D\) by the map \(f_i\). The equations above give the defining equations of \(\tilde{X}_\lambda\) at \(q_1\) and \(q_2\), respectively.

Concerning the equation (13), if \(B_1^*(0) \neq 0\) or \(D_1^*(0) \neq 0\), then \(\tilde{X}_\lambda\) is non-singular at \(q_1\). Assume \(B_1^*(0) = D_1^*(0) = 0\) to the contrary, then \(B(0) = D(0) = 0\).

\[
A(0)x + B(0)y + C(0)z + D(0)w = 0
\]
is the equation of the embedded tangent space to \(H_\lambda\) at \(p\) in terms of the local coordinate \((x, y, z, w)\), and since \(H_\lambda\) is defined by the equation (4), we have
\[
\sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) = B(0) = 0, \quad \text{and} \quad \sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) = D(0) = 0.
\]
On the other hand, since \(\lambda \not\in B\), this is because of the condition (10), we have
\[
\sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) \neq 0, \quad \text{or} \quad \sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) \neq 0.
\]
This is a contradiction. Therefore we conclude that \(B_1^*(0) \neq 0\) or \(D_1^*(0) \neq 0\), and so \(\tilde{X}_\lambda\) is non-singular at \(q_1\). Concerning the equation (14), if \(A_2^*(0) = B_2^*(0) = C_2^*(0) = 0\), then \(A(0) = B(0) = C(0) = 0\). This means the equation of the embedded tangent space to \(H_\lambda\) at \(p\) with respect to the local coordinate \((x, y, z, w)\) is \(w = 0\), that is, \(H_\lambda\) is tangent to the hypersurface \(w = 0\) at \(p\). But this is a contradiction, because, since \(\lambda \not\in X^\vee\), \(H_\lambda\) is not a limit of tangent hyperplanes to \(X\) in \(P^4(C)\) at simple points of \(X\). Therefore we conclude that at least one of \(A_2^*(0), B_2^*(0)\) and \(C_2^*(0)\) is not zero, and so \(\tilde{X}_\lambda\) is non-singular at \(q_2\).

The assertion (ii): From the same reasoning as in the proof of the assertion (i) it follows that \(\tilde{X}_\lambda\) is non-singular at every point of \(f^{-1}(D_{\lambda\epsilon})\) where \(D_{\lambda\epsilon} = X_{\lambda\epsilon} \cap D\). Hence it suffices to show that \(X_{\lambda\epsilon}\) has only one isolated ordinary double point on \(X_{\lambda\epsilon} \cap X_{sm}\). By the manner of choosing the line \(L^*\) in \(P^4(C)^\vee\), the hyperplane \(H_{\lambda\epsilon}\) is tangent to \(X\) at only one point, say \(q\), of \(X_{sm}\). Therefore \(X_{\lambda\epsilon}\) is non-singular at all but one point \(q\) of \(X_{\lambda\epsilon} \cap X_{sm}\). To prove that \(X_{\lambda\epsilon}\) has an isolated ordinary double point at \(q\), we assume that the homogeneous coordinate \([x_0 : x_1 : x_2 : x_3 : x_4]\) of \(q\) is \([1 : 0 : 0 : 0 : 0]\) and \(H_{\lambda\epsilon}\) is defined by \(x_4 = 0\). We put \(\zeta_i = x_i / x_0 \ (1 \leq i \leq 4)\), and use this linear affine coordinate \((\zeta_1, \ldots, \zeta_4)\) in the subsequent arguments. Then \(X\) is defined by \(F(1, \zeta_1, \zeta_2, \zeta_3, \zeta_4) = 0\), \(q\) is the origin \((0, \ldots, 0)\), and \(H_{\lambda\epsilon}\) is defined by \(\zeta_4 = 0\). Since the tangent hyperplane to \(X\) at \(q\) is the hyperplane \(H_{\lambda\epsilon} : \zeta_4 = 0\), we have
\[
\frac{\partial F}{\partial \zeta_i}(1, 0, \ldots, 0) = 0 \quad (1 \leq i \leq 3)
\]
\[
\frac{\partial F}{\partial \zeta_4}(1, 0, \ldots, 0) \neq 0.
\]
Because of (16), there is an analytic function \( \phi(\zeta_1, \zeta_2, \zeta_3) \) of the variables \( \zeta_1, \zeta_2, \zeta_3 \) defined in a neighborhood of the origin, which satisfies the following:

\[
\begin{align*}
(17) & \quad \phi(0, 0, 0) = 0, \\
(18) & \quad F(1, \zeta_1, \zeta_2, \zeta_3, \phi(\zeta_1, \zeta_2, \zeta_3)) \equiv 0 \pmod{\text{locally}}.
\end{align*}
\]

This means that the defining equation of \( X \) in a neighborhood of \( q \) is given by

\[
(19) \quad \zeta_4 = \phi(\zeta_1, \zeta_2, \zeta_3)
\]

By the same reasoning as before, we have

\[
(20) \quad \frac{\partial \phi}{\partial \zeta_i}(0, 0, 0) = 0 \quad (1 \leq i \leq 3)
\]

Hence \( \phi \) is expressed as

\[
(21) \quad \phi = \sum_{1 \leq i, j \leq 3} \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j}(0) \zeta_i \zeta_j + O(3)
\]

If we regard \( (\zeta_1, \zeta_2, \zeta_3) \) as a local coordinate on \( H_\lambda \), \( X_\lambda \) is defined by \( \phi(\zeta_1, \zeta_2, \zeta_3) = 0 \) in \( H_\lambda \). Therefore, if we prove

\[
(22) \quad \det \left( \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j}(0) \right) \neq 0
\]

then we can conclude that, after suitable change of local coordinates, the defining equation of \( X_\lambda \) will become

\[
\phi(\zeta_1, \zeta_2, \zeta_3)(\zeta_1^2 + \zeta_2^2 + \zeta_3^2) = 0
\]

in a neighborhood of the origin in \( H_\lambda \), where \( u(\zeta_1, \zeta_2, \zeta_3) \) is a non-vanishing analytic function. This proves the assertion (ii) holds. To prove (22), we evaluate the Hessian

\[
\det \left( \frac{\partial^2 F}{\partial x_i \partial x_j}(1, 0) \right)
\]

If \( x_0 \neq 0 \), by use of (24) and (23), we can derive

\[
(25) \quad \det \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right) = \left( \frac{n-1}{x_0} \right)^2 \left| \begin{array}{cccc}
\frac{n}{n-1} F & \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_4} \\
\frac{\partial F}{\partial x_1} & \frac{\partial^2 F}{\partial x_1^2} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_4} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F}{\partial x_4} & \frac{\partial^2 F}{\partial x_4 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_4^2}
\end{array} \right|
\]

First we mention some remarks about \( \det(\partial^2 F/\partial x_i \partial x_j) \) of the homogeneous polynomial \( F \) at \( q = [1 : 0 : 0 : 0] \).

From the Euler identity

\[
\sum_{i=0}^{n} x_i \frac{\partial F}{\partial x_i} = n F \quad (n = \deg F),
\]

it follows that

\[
(24) \quad \sum_{j=0}^{n-1} y_j \frac{\partial^2 F}{\partial x_i \partial x_j} = (n - 1) \frac{\partial F}{\partial x_i} \quad (0 \leq i \leq 4).
\]

If \( x_0 \neq 0 \), by use of (24) and (23), we can derive

\[
(25) \quad \det \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right) = \left( \frac{n-1}{x_0} \right)^2 \left| \begin{array}{cccc}
\frac{n}{n-1} F & \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_4} \\
\frac{\partial F}{\partial x_1} & \frac{\partial^2 F}{\partial x_1^2} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_4} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F}{\partial x_4} & \frac{\partial^2 F}{\partial x_4 \partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_4^2}
\end{array} \right|
\]
Therefore, since \( F(1,0) = 0 \) and \( (\partial F/\partial x_i)(1,0) = 0 \) \((1 \leq i \leq 3)\) (cf. (15)), we have
\[
(26) \quad \det\left(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\right) = (n - 1)^2 \left(\frac{\partial F}{\partial x_4}(1,0)\right)^2 \det\left(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\right)_{1 \leq i,j \leq 3}
\]
Here we need to recall that \( \Phi(q) = \lambda \) does not belong to \((X^v)^{\text{sing}}\) because of the condition (9). This means the Gauss map \( \Phi \) defined by (1) gives a biregular morphism between \( X \) and \( X^v \) in a neighborhood of \( q \). Therefore the right-hand-side of (26) is not zero, and so we have
\[
(27) \quad \det\left(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\right)_{1 \leq i,j \leq 3} \neq 0
\]
since \( (\partial F/\partial x_4)(1,0) \neq 0 \) (cf. (16)). On the other hand, derivating the equation (18) twice with respect to the variables \( \zeta_1, \zeta_2, \zeta_3 \) and substituting 0 for all \( \zeta_i \), we have
\[
(28) \quad \det\left(\frac{\partial^2 F}{\partial x_i x_j}(1,0)\right)_{1 \leq i,j \leq 3} = -\left(\frac{\partial F}{\partial x_4}(1,0)\right)^3 \det\left(\frac{\partial^2 \phi}{\partial \zeta_i \zeta_j}(1,0)\right)
\]
Since \( (\partial F/\partial x_4)(1,0) \neq 0 \), by (28) and (27) we have
\[
\det\left(\frac{\partial^2 \phi}{\partial \zeta_i \zeta_j}(1,0)\right) \neq 0
\]
as required. This completes the proof of the theorem. ■

In what follows we assume that \( P_\infty \) is sufficiently general so that Theorem 2.1 holds.

**Lemma 2.2** With the notation in Proposition 2.1, we have the following:

(i) \( \overline{C_\infty} := f^{-1}(C_\infty) \) is a non-singular curve,
(ii) \( \mathcal{L} := \bigcup_{\lambda \in P} X_\lambda \) is a linear system on \( X_1 \) with the base point locus \( B_s(\mathcal{L}) = \overline{C_\infty} \), and
(iii) for \( \lambda, \mu \in P \) with \( \lambda \neq \mu \), \( X_\lambda \) and \( X_\mu \) intersect transversely along \( \overline{C_\infty} \).

**Proof.** We take an affine coordinate neighborhood \( U \) of \( P^4(\mathbb{C}) \) with \( U \cap P_\infty \neq \emptyset \), and work on this neighborhood. Let \((\zeta_1, \zeta_2, \zeta_3, \zeta_4)\) be a linear affine coordinate on \( U \). We may assume that
\[
(29) \quad (a) \quad P_\infty \cap T = \emptyset \quad \text{and} \quad P_\infty \cap C = \emptyset,
(b) \quad P_\infty \quad \text{and} \quad X \text{ intersect transversely at every non-singular point of} \ X, \quad \text{and}
(c) \quad P_\infty \quad \text{and} \quad D \text{ intersect transversely.}
\]
Let \( P_\infty = H_0 \cap H_1 \) where \( H_0 \) and \( H_1 \) are hyperplanes in \( P^4(\mathbb{C}) \), and let \( \varphi_i \) be a linear function which defines \( H_i \) on \( U \) for \( i = 1, 2 \). Note that the linear system \( \mathcal{L} := \bigcup_{\lambda \in P} X_\lambda \) is defined by \( \alpha f^* \varphi_0 + \beta f^* \varphi_1 \) \((\alpha, \beta \in \mathbb{C})\) where \( f^* \varphi_i \) \((i = 1, 2)\) denotes the pull-back of \( \varphi_i \) by the normalization map \( f : X_1 \rightarrow X \). Therefore the assertion (ii) is trivial. By the assumption (29), (b), the assertions (i) and (iii) also trivially hold at \( q = f^{-1}(p) \) for a non-singular point \( p \) of \( X \), so we will prove that the assertions (i) and (iii) hold at \( q \in f^{-1}(p) \) for \( p \in D \cap U \). We assume that \( X \) is defined by \( XY = 0 \) with respect to some complex analytic local coordinate \((X, Y, Z, W)\) with center \( p \), and assume that the normalization map \( f \) is given by
\[
(u, v, t) \rightarrow (0, u, v, t) = (X, Y, Z, W),
\]
where \((u,v,t)\) is a complex analytic local coordinate with center \(q := f^{-1}(p)\). The Jacobian matrix of \(f^*\varphi_0, f^*\varphi_1\) with respect to \((u,v,t)\) at \(q\) is given as follows:

\[
\frac{\partial(f^*\varphi_0, f^*\varphi_1)}{\partial(u,v,t)}(q)
\]

\[
= \left(\begin{array}{c}
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial Y} \frac{\partial \varphi_0(p)}{\partial \zeta_i(p)}, \\
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial Z} \frac{\partial \varphi_0(p)}{\partial \zeta_i(p)}, \\
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial W} \frac{\partial \varphi_0(p)}{\partial \zeta_i(p)}
\end{array}\right)
\left(\begin{array}{c}
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial Y} \frac{\partial \varphi_1(p)}{\partial \zeta_i(p)}, \\
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial Z} \frac{\partial \varphi_1(p)}{\partial \zeta_i(p)}, \\
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial W} \frac{\partial \varphi_1(p)}{\partial \zeta_i(p)}
\end{array}\right)
\]

On the other hand, by the assumption (29), (c),

\[
\left| \begin{array}{cc}
\frac{\partial \varphi_0(p)}{\partial Y} & \frac{\partial \varphi_0(p)}{\partial W}
\end{array} \right| \neq 0.
\]

Hence,

\[
\left| \begin{array}{cc}
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial Y} \frac{\partial \varphi_0(p)}{\partial \zeta_i(p)}, \\
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial Z} \frac{\partial \varphi_0(p)}{\partial \zeta_i(p)}, \\
\sum_{i=1}^{4} \frac{\partial\zeta_i(p)}{\partial W} \frac{\partial \varphi_0(p)}{\partial \zeta_i(p)}
\end{array} \right| \neq 0.
\]

By (30) and (31), we conclude \(\{\partial(f^*\varphi_0, f^*\varphi_1)/\partial(u,v,t)\}(p)\) has the maximal rank. From this it follows that \(\widehat{C}_\infty\) is non-singular at \(q\). Furthermore, if \([\alpha : \beta] \neq [\alpha' : \beta']\) as a point of \(P^1(C)\), then \(\alpha\beta' - \alpha'\beta \neq 0\), so

\[
\frac{\partial(f^*\varphi_0, f^*\varphi_1)}{\partial(u,v,t)}(q) \quad \text{and} \quad \frac{\partial(\alpha f^*\varphi_0 + \beta f^*\varphi_1, \alpha' f^*\varphi_0 + \beta' f^*\varphi_1)}{\partial(u,v,t)}(q)
\]

have the same rank. Hence \(\{\partial(\alpha f^*\varphi_0 + \beta f^*\varphi_1, \alpha' f^*\varphi_0 + \beta' f^*\varphi_1)/\partial(u,v,t)\}(q)\) has also the maximal rank. This shows that the assertion (iii) holds at \(q\) as required. This completes the proof of the lemma. ■

Let \(\sigma : \widehat{X}_1 \to X_1\) be the blowing-up along \(\widehat{C}_\infty := f^{-1}(C_\infty)\), and \(\widehat{L} := \bigcup_{\lambda \in P} \widehat{X}_\lambda\) the proper inverse of \(\widehat{L} := \bigcup_{\lambda \in P} \widehat{X}_\lambda\). Then \(\widehat{L}\) gives a fibering of \(\widehat{X}_1\) over \(P \simeq P^1(C)\), which we denote by \(\pi : \widehat{X}_1 \to P\). Let \(S = \{\lambda_1, \cdots, \lambda_c\}\) and \(\widehat{X}_1^* = \widehat{X}_1 - \pi^{-1}(S)\). From the exact Borel-Moore homology sequence determined by the space, the closed subspace, and its complement, it follows that

\[
\chi(\widehat{X}_1) = \chi(\widehat{X}_1^*) + \chi(\pi^{-1}(S)).
\]
It is clear that
\[ \chi(S_{\pi - 1}(S)) = \sum_{i=1}^{c} \chi(\tilde{X}_{\lambda_i}). \]  

(33)

Since $\tilde{X}_{1}^* \to P - S$ is locally trivial (as a differential fiber space), it follows from the spectral sequence of Leray for this fiber space that
\[ \chi(\tilde{X}_{1}^*) = \chi(\tilde{X}_\lambda)\chi(P - S), \]

(34)

where $\tilde{X}_\lambda$ denote a generic fiber of $\tilde{X}_{1}^* \to P - S$. By the same reason as before, we have
\[ \chi(P) = \chi(P - S) + c. \]  

(35)

Comparing (32), (33), (34) and (35), we have
\[ \chi(\tilde{X}_1) = \chi(P^1(C))\chi(\tilde{X}_\lambda) + \sum_{j=1}^{c} (\chi(\tilde{X}_{\lambda_j}) - \chi(\tilde{X}_\lambda)) + c = 2\chi(\tilde{X}_\lambda) - c. \]

The second equality above follows from the fact that a topological 2-cycle vanishes when $\lambda \to \lambda_j$ for $j = 1, \cdots, c$. We put $\tilde{E} := \sigma^{-1}(\tilde{C}_\infty)$. Then, since $\tilde{X}_1 \setminus \tilde{E} \approx X_1 \setminus \tilde{C}_\infty$,
\[ \chi(\tilde{X}_1) - \chi(X_1) = \chi(\tilde{E}) - \chi(\tilde{C}_\infty) \]
\[ = \chi(P^1(C))\chi(\tilde{C}_\infty) - \chi(\tilde{C}_\infty) \]
\[ = 2\chi(\tilde{X}_\lambda) - c. \]

Hence,
\[ \chi(X_1) = \chi(\tilde{X}_1) - \chi(\tilde{C}_\infty) = 2\chi(\tilde{X}_\lambda) - \chi(\tilde{C}_\infty) - c = 2\chi(\tilde{X}_\lambda) - \chi(\tilde{C}_\infty) - c. \]

(36)

Since $C_\infty$ is a curve whose degree is equal to $n$ with $m$ ordinary double points in $P_\infty \simeq P^2(C)$, the genus $g(C_\infty)$ is given by
\[ g(C_\infty) = \frac{1}{2}(n - 1)(n - 2) - m. \]

Hence,
\[ \chi(C_\infty) = 2 - 2g(C_\infty) = 2 - (n - 1)(n - 2) + 2m. \]

(37)

Note that $X_\lambda$ is a surface with ordinary singularities in $H_\lambda \simeq P^2(C)$ of degree $n$, whose numerical characteristics related to its singularities are as follows:

- the degree of its double curve $D_\lambda = m$
- $\#\{\text{triple points of } X_\lambda\} = t$
- $\#\{\text{cuspidal points of } X_\lambda\} = \gamma$.

Therefore, by the classical formula,
\[ \chi(\tilde{X}_\lambda) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma \]

(38)

By (36), (37) and (38), we have the following:

**Proposition 2.3**
\[ \chi(X_1) = 2n(n^2 - 4n + 6) - 2(3n - 8)m + 6t - 4\gamma \]
\[ -2 + (n - 1)(n - 2) - 2m - c \]
\[ = n(2n^2 - 7n + 9) - 2(3n - 7)m + 6t - 4c - c \]

3. The computation of the class of an algebraic threefold with ordinary singularities in \( P^4(\mathbb{C}) \). Throughout this section we denote a rational equivalence class of an algebraic cycle, say \( \alpha \), by \( [\alpha] \), and denote the intersection class of two algebraic cycle classes, say \([\alpha]\) and \([\beta]\), by \( \alpha \cdot \beta \). We refer to the following theorem from [5].

**Theorem 3.1** ([5], Theorem (2.3)) Let \( X^n \) be a hypersurface of degree \( d \) in \( P^{n+1} \). Then its \( k \)-th polar class is given by
\[
[M_k] = [(d - 1)c_1(L)]^k \cap [X] - \sum_{i=0}^{k-1} \binom{k}{i} [(d - 1)c_1(L)]^i \cap s_{n-i}(J, X) \quad (0 \leq k \leq n)
\]
where \( L = \mathcal{O}_{P^n}(1) \) and \( s(J, X) = \sum_{k=0}^n s_k(J, X) \) \( (s_k(J, X) \in A_k(J)) \) denotes the Segre class of the singular subscheme \( J \) of \( X \).

In what follows, using the theorem above, we shall compute the class \( c \) of an algebraic threefold with ordinary singularities in the complex projective 4-space \( P^4(\mathbb{C}) \) for the case where the threefold is free from quadruple points. First we fix the notation as follows:

\( Y = P^4(\mathbb{C}) \): the complex projective 4-space,
\( \overline{X} \): an algebraic threefold with ordinary singularities in \( Y \), which is free from quadruple points,
\( \overline{J} \): the singular subscheme of \( \overline{X} \) defined by the Jacobian ideal of \( \overline{X} \),
\( \overline{D} \): the singular locus of \( \overline{X} \),
\( \overline{T} \): the triple point locus of \( \overline{X} \), which is equal to the singular locus of \( \overline{D} \),
\( \overline{C} \): the cuspidal point locus of \( \overline{X} \), precisely, its closure, since we always consider \( \overline{C} \)
contains the stationary points,
\( \Sigma_{\overline{C}} \): the stationary point locus of \( \overline{X} \),
\( n_{\overline{X}} : X \to \overline{X} \): the normalization of \( \overline{X} \),
\( f : X \to Y \): the composite of the normalization map \( n_{\overline{X}} \) and the inclusion \( \iota : \overline{X} \hookrightarrow Y \),
\( J \): the scheme-theoretic inverse of \( \overline{J} \) by \( f \),
\( D, T, C \) and \( \Sigma f \): the inverse images of \( \overline{D}, \overline{T}, \overline{C} \) and \( \Sigma_{\overline{C}} \) by \( f \), respectively.

Note that \( \overline{T} \) and \( \overline{C} \) are non-singular curves, intersecting transversely at \( \Sigma_{\overline{C}} \), and that the normalization \( X \) of \( \overline{X} \) is also non-singular. Calculating by use of local coordinates, we can easily see the following:

(i) \( J \) contains \( D \), and the residual scheme (cf. [2], Definition 9.2.1, p. 160) to \( D \) in \( J \)
is the reduced scheme \( C \);
(ii) \( D \) is a surface with ordinary singularities, free from triple points, whose singular locus is \( T \),
(iii) \( D \) is the double point locus of the map \( f : X \to Y \), i.e., the closure of \( \{ q \in X \mid \# f^{-1}(f(q)) \geq 2 \} \);
(iv) the map \( f_{|D} : D \to \overline{D} \) is generically two to one, simply ramified at \( C \);
(v) the map \( f_{|T} : T \to \overline{T} \) is generically three to one, simply ramified at \( \Sigma_{\overline{C}} \).

To compute the Segre class \( s(J, X) \), the following proposition is useful.
Proposition 3.2 ([2], Proposition 9.2, p. 161) Let $D \subset W \subset V$ be closed embeddings of schemes, with $V$ a $k$-dimensional variety, and $D$ a Cartier divisor on $V$. Let $R$ be the residual scheme to $D$ in $W$. Then, for all $m$,

$$s(W, V)_m = s(D, V)_m + \sum_{j=0}^{k-m} \binom{k-m}{j} [-D]^j \cdot s(R, V)_{m+j}$$

in $A_m(W)$, the $m$-th rational equivalence class group of algebraic cycles on $W$.

In our case, since $D = f^{-1}(\mathcal{D})$ is a Cartier divisor, its normal cone $\mathcal{C}_D X$ to $D$ in $X$ is isomorphic to $\mathcal{O}_X(D)_{|D}$, the restriction to $D$ of the line bundle $\mathcal{O}_X(D)$ associated to $D$. Therefore,

$$s(D, X) = c(\mathcal{O}_X(D)_{|D})^{-1} \cap [D] = [D] - c_1(\mathcal{O}_X(D)_{|D}) \cap [D] + c_1(\mathcal{O}_X(D)_{|D})^2 \cap [D] = [D] - [D]^2 + [D]^3.$$  

Since $C$ is non-singular,

$$c(N_{C/X})^{-1} \cap [C] = [C] - c_1(N_{C/X}) \cap [C].$$

Hence, applying Proposition 3.2 for $W = J$, $D_f = f^{-1}(\mathcal{D})$ and $R = C$, we have

$$\begin{align*}
\left\{\begin{array}{l}
s(J, X)_1 = [D] \\
J
\end{array}\right.
\begin{array}{l}
s(J, X)_0 = -[D]^2 + [C] \\
J
\end{array}\right.
\begin{array}{l}
s(J, X)_0 = [D]^3 - c_1(N_{C/X}) \cap [C] - 3D \cdot C \\
J
\end{array}\right.
\begin{array}{l}
s(\mathcal{J}, \mathcal{X})_2 = f_\ast s(J, X)_2, \text{ from the first equality above, it follows that} \\
J
\end{array}\right.
\begin{array}{l}
s(\mathcal{J}, \mathcal{X})_2 = 2[D] \\
J
\end{array}\right.
\begin{array}{l}
\text{To know } s(\mathcal{J}, \mathcal{X})_1, \text{ we need to understand } f_\ast[D]^2, \text{ the push-forward of } [D]^2 \text{ by } f. \text{ For this purpose, we compute } f_\ast[D]^2. \text{ To do this, we consider the following fiber square:}
\end{array}\right.
\begin{array}{l}
X' \xrightarrow{f'} Y' \\
\tau_t \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \sigma_t \\
X \xrightarrow{f} Y.
\end{array}\right.$$

Here

$\sigma_t : Y' \rightarrow Y$ : the blowing-up of $Y$ along the triple point locus $T$ of $\mathcal{X}$,

$\mathcal{X}'$ : the proper inverse image of $\mathcal{X}$ by $\sigma_T$,

$X' := X \times_{\mathcal{X}} \mathcal{X}'$ : the fiber product of $X$ and $\mathcal{X}'$ over $\mathcal{X}$,

$\nu_{\mathcal{X}} : X' \rightarrow \mathcal{X}'$ : the projection to the second factor of $X \times_{\mathcal{X}} \mathcal{X}'$, which is nothing but the normalization of $\mathcal{X}'$,

$f' : X' \rightarrow Y' :$ the composite of the normalization map $\nu_{\mathcal{X}}$ and the inclusion $' : \mathcal{X}' \hookrightarrow Y'$,

$\tau_t : X' \rightarrow X$ : the projection to the first factor of $X \times_{\mathcal{X}} \mathcal{X}'$, which is nothing but the blowing-up of $X$ along $T$.

In what follows, we denote by $\mathcal{D}'$, $\mathcal{T}'$ and $\mathcal{C}'$ the proper inverse images of $\mathcal{D}$, $T$ and $\mathcal{O}$ by $\sigma_T$, respectively. We consider the following fiber square:
where $E_T = P(N_T Y)$ is the exceptional divisor of the blowing-up $\sigma_T$, which is a $P^2(\mathbb{C})$-bundle on $T$, and $p : E_T \rightarrow T$ is the projection to the base space of this bundle. We denote by $\mathcal{O}_{N_T Y}(1)$ the canonical line bundle on $E_T$. Then the tautological line bundle on $E_T$ is $\mathcal{O}_{N_T Y}(-1)$, which is a subbundle of $p^* N_T Y$.

**Lemma 3.3** $\sigma_T^*[D]$ is expressed as

$$\sigma_T^*[D] = [D] + 3[p_* [\xi_T]] + [p^* [\alpha_0]]$$

where $[\xi_T] = c_1(\mathcal{O}_{N_T Y}(1)) \cap [E_T]$ and $[\alpha_0]$ an algebraic 0-cycle class on $T$.

**Proof.** By the blow-up formula([2], Theorem 6.7, p.116),

$$\sigma_T^*[D] = [D] + [c(E) \cap p^* s(T, D)]_2$$

where $E = p^* N_T Y/N_{E_T} Y' = p^* N_T Y/\mathcal{O}_{N_T Y}(-1)$. Since

$$c_1(E) = p^* c_1(N_T Y) - c_1(\mathcal{O}_{N_T Y}(-1)) = p^* c_1(N_T Y) + c_1(\mathcal{O}_{N_T Y}(1)),$$

we have

$$\{c(E) \cap s(T, D)\}_2 = p^* s_0(T, D) + c_1(E) \cap p^* s_1(T, D)$$

$$= p^* \{s_0(T, D) + c_1(N_T Y) \cap s_1(T, D)\} + c_1(\mathcal{O}_{N_T Y}(1)) \cap p^* s_1(T, D)$$

To compute $s(T, D)$, we consider the normalization map $n_{D^*} : D^* \rightarrow D$. $D^*$ is non-singular. Hence, if we put $T^* := n_{D^*}^{-1}(T)$, we have

$$s(T, D^*) = c(N_T D^*)^{-1} \cap [T^*]$$

$$= (1 - c_1(N_T D^*)) \cap [T^*]$$

$$= [T^*] - [T^* \cdot T^*].$$

Therefore,

$$s(T, D) = n_{D^*} s(T^*, D^*) = 3[T] - n_{D^*} (T^* \cdot T^*),$$

and so,

$$\left\{ \begin{array}{l}
    s_0(T, D) = -n_{D^*} (T^* \cdot T^*) \\
    s_1(T, D) = 3[T]
\end{array} \right.$$  

By (45) and (46), if we put $[\alpha_0] := -n_{D^*} (T^* \cdot T^*) + 3c_1(N_T Y) \cap [T]$,

$$\{c(E) \cap s(T, D)\}_2 = p^*[\alpha_0] + 3[\xi_T].$$

Consequently, by (44), we have the equality in (43). $lacksquare$

**Proposition 3.4**

$$[D]^2 = f^*[X] \cdot D - f^*[D] + [T] - [C]$$
PROOF. To know $|D|^2$, we compute $f^*\mathcal{D}$. For this purpose, we use the diagram in (41). Since $\tau_r : X' \to X$ is a blowing-up, we have $\tau_r^* \tau_r^* \alpha = \alpha$ for any algebraic cycle $\alpha \in A_*(X)$. Hence,

$$(48) \quad \tau_r^* f^* \mathcal{D} = \tau_r^* \tau_r^* f^* \mathcal{D} = f^* \mathcal{D}.$$

Since $\mathcal{D}$ is regularly embedded in $Y'$, i.e., $C_{\mathcal{D}/Y'} \simeq N_{\mathcal{D}/Y'}$, while $\mathcal{D}$ is not, we can apply the excess intersection formula ([2, Theorem 6.3, p.102]) to $\mathcal{D}$. Then, denoting the tangent bundle of a non-singular algebraic variety, say $Z$, by $T_Z$ we have

$$(49) \quad f^*\mathcal{D} = c_1(f^*N_{\mathcal{D}/Y'}) - c_1(T_{\mathcal{D}/Y'}) \cap [D']$$

$$= \{c_1(f^*T_{Y'}) - c_1(T_{Y'})\} \cap [D']$$

$$= \{c_1(f^*T_{Y'}) - c_1(T_{X'})\} \cap [D'] - C',$$

where the last equality follows from the ramification formula ([2, Example 3.2.20, p.62]).

On the other hand, by the double point formula ([2, Theorem 9.3, p.166]),

$$(50) \quad [D'] = f^*[X] - \{c_1(f^*T_{Y'}) - c_1(T_{X'})\} \cap [X'].$$

By (49) and (50), we have

$$(51) \quad f^*\mathcal{D} = f^*[X] \cdot D' - [D']^2 - C'.$$

Next, in view of Lemma 3.3, we compute $f^*[3j_*[\xi_T] + j_*\mathcal{D}[\alpha_0]]$. For this purpose, we consider the following fiber square:

$$(52) \quad \begin{array}{ccc}
E_T & \rightarrow & X' \\
\downarrow j & & \downarrow \tau_r \\
T & \rightarrow & X,
\end{array}$$

where $E_T = P(N_{T/X})$ is the exceptional divisor of the blowing-up $\tau_T$, which is a $P^1(C)$-bundle on $T$, and $p : E_T \to T$ is the projection to the base space of this bundle. There is a set of morphisms from the diagram in (52) to the one in (42) induced by those in the diagram in (41). We denote by $g$ and $g'$ the restriction of $f : X \to Y$ to $T$ and that of $f' : X' \to Y'$ to $E_T$, respectively. Note that the morphism $g' : E_T \to E_T$ maps each fiber of $p : E_T \to T$ to that of $\mathcal{D} : E_T \to \mathcal{T}$, and so $g^*\mathcal{D} = [\xi_T]$ where $\xi_T = c_1(\mathcal{O}_{N_{T/X}}(1)) \cap [E_T]$. Since $f' : X' \to Y'$ and $g' : E_T \to E_T$ are local complete intersection morphisms of the same codimension, we can apply Proposition 6.6, (c) in [2, p.113] to the fiber square

$$(53) \quad \begin{array}{ccc}
E_T & \rightarrow & E_T \\
\downarrow j & & \downarrow j \\
X' & \rightarrow & Y'.
\end{array}$$

Then, $f^*\mathcal{D} = j_*g^*[\xi_T]$ and $f^*\mathcal{D}[\alpha_0] = j_*g^*\mathcal{D}[\alpha_0] = j_*p^*g^*[\alpha_0]$. Therefore, we have

$$(54) \quad f^*[3j_*[\xi_T] + j_*\mathcal{D}[\alpha_0]] = 3j_*[\xi_T] + j_*\mathcal{D}[\alpha_0].$$

By (43), (51) and (54), we have

$$f^*\mathcal{D} = f^*[X] \cdot D' - [D']^2 - C' + 3j_*[\xi_T] + j_*\mathcal{D}[\alpha_0].$$
Since \( \tau_\star[C'] = [C], \tau_\star.j_\star[\xi_T] = T \) and \( \tau_\star.j_\star.g^*\alpha_0 = 0 \), by the equality above and (48),

\[
f^*[D] = \tau_\star.f^*\sigma_T[D] \tag{55}
\]

Since \( \tau_T[D] = [D'] + 2[E_T], \)

\[
\tau_\star(f^{**}[X] \cdot D') = \tau_\star(f^{**}[X] \cdot \tau_T^*[D] - 2f^{**}[X] \cdot E_T) \tag{56}
\]

On the other hand, since \( \sigma_T^*[X] = [X] + 3[E_T], \)

\[
f^{**}[X] = f^*\sigma_T^*[X] - 3[E_T].
\]

Hence, by the projection formula,

\[
\tau_\star(f^{**}[X] \cdot \tau_T^*[D]) = \tau_\star(f^{**}[X]) \cdot D \tag{57}
\]

and

\[
\tau_\star(f^{**}[X] \cdot E_T) = \tau_\star(f^{**}\sigma_T^*[X] \cdot E_T - 3[E_T]^2) \tag{58}
\]

Therefore, by (56), (57) and (58),

\[
\tau_\star(f^{**}[X] \cdot D') = f^*[X] \cdot D - 6[T]. \tag{59}
\]

Furthermore, we have

\[
\tau_\star[D]^2 = \tau_\star((\tau_T^*[D] - 2[E_T])^2) \tag{60}
\]

Consequently, by (55), (59) and (60),

\[
f^*[D] = f^*[X] \cdot D - 6[T] - [D]^2 + 4[T] - [C] + 3[T] = f^*[X] \cdot D - [D]^2 - [C] + [T],
\]

from which the equality (47) follows. \( \blacksquare \)

Since \( f_\star[X] = [X], f_\star[D] = 2[D], f_\star[T] = 3[T] \) and \( f_\star[C] = [C] \), by Proposition 3.4, we have the following:

**Corollary 3.5**

\[
f_\star[D]^2 = \overline{X} \cdot \overline{D} + 3[T] - [\overline{C}] \tag{61}
\]

By Proposition 3.4 and the second equality in (39),

\[
s(J, X)_1 = -f^*[X] \cdot D + f^*[D] - [T] + 2[C]
\]
and so, by the projection formula

\[(62) \quad s(J, X)_1 = -X \cdot D - 3[T] + 2[\overline{C}]\]

Now we compute \(s(J, X)_0\). By Proposition 3.4, \(\[D]\)^3 = f^*[X] \cdot [D]^2 - f^*[\overline{D}] \cdot D + D \cdot T - D \cdot C\)

Hence, by the third equality in (39),

\[(63) \quad s(J, X)_0 = f^*[X] \cdot [D]^2 - f^*[\overline{D}] \cdot D + D \cdot T - 4D \cdot C - c_1(N_X \cap C)\]

Since \(T\) and \(\overline{C}\) are regularly embedded in \(Y\), we can apply the excess intersection formula to them. Then,

\[
f^*[T] = c_1(f^*N_{T/Y}/N_TX) \cap [T]
\]

\[
= \{c_1(f^*T_Y) - c_1(f^*T_T) - c_1(T_X)\} \cap [T]
\]

\[
= \{c_1(f^*T_Y) - c_1(T_X)\} \cap [T] - [\Sigma s]
\]

\[
= f^*[X] \cdot T - D \cdot T - [\Sigma s],
\]

where the last step but one follows from the ramification formula for \(g : T \rightarrow T\) and the last step from the double point formula for \(f : X \rightarrow Y\). Similarly, since \(\overline{C} \simeq C\), we have

\[
f^*[C] = f^*[X] \cdot C - D \cdot C
\]

Therefore we have

\[(64) \quad \left\{ \begin{aligned}
D \cdot T &= f^*[X] \cdot T - f^*[T] - [\Sigma s] \\
D \cdot C &= f^*[X] \cdot C - f^*[\overline{C}]
\end{aligned} \right.\]

By the adjunction formula, the double point formula for \(f : X \rightarrow Y\) and the second equality in (64),

\[(65) \quad c_1(N_X \cap C) = -K_X \cdot C + [k_C]
\]

\[
= (-f^*[X] + K_Y) \cdot C + [k_C]
\]

\[
= -f^*[K_Y] \cdot C - f^*[\overline{C}] + [k_C],
\]

where \(K_Y, K_X\) and \(k_C\) are the canonical divisors of \(Y, X\) and \(C\), respectively. Substituting (64) and (65) into (63), we have

\[s(J, X)_0 = f^*[X] \cdot [D]^2 - f^*[\overline{D}] \cdot D + f^*[X] \cdot T - f^*[T] - [\Sigma s]
\]

\[-4f^*[X] \cdot C + 4f^*[\overline{C}] + f^*[K_Y] \cdot C + f^*[\overline{C}] - [k_C].\]

Consequently, using Corollary 3.5 and the fact that \(f_*[X] = [X], f_*[D] = 2[\overline{D}], f_*[T] = 3[T], f_*[\Sigma s] = [\Sigma s]\) and \(\overline{C} \simeq C\), we have,

\[s(J, X)_0 = [X]^2 \cdot 3[\overline{D}]^2 + 5X \cdot T + K_Y \cdot \overline{C} - [k_C] - [\Sigma s].\]

We collect the results obtained till now in the following proposition:

**Proposition 3.6** The Segre classes of the singular subscheme \(J\), defined by the Jacobian ideal, of an algebraic threefold \(X\) with ordinary singularities in the four dimensional
projective space $Y = P^4(C)$ are given as follows, if $\overline{X}$ is free from quadruple points:

$$
\begin{align*}
\left\{ \begin{array}{l}
s(\overline{J}, \overline{X})_2 = 2[\overline{D}] \\
s(\overline{J}, \overline{X})_1 = -\overline{X} \cdot \overline{D} - 3\overline{T} + 2\overline{C} \\
s(\overline{J}, \overline{X})_0 = [\overline{X}]^2 \cdot \overline{D} - 2[\overline{D}]^2 + 5\overline{X} \cdot \overline{T} + K_Y \cdot \overline{C} - [k_{\overline{C}}] - [\Sigma \overline{C}]
\end{array} \right.
\end{align*}
$$

Here $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\Sigma \overline{C}$ are the singular locus, triple point locus, cuspidal point locus and stationary point locus of $\overline{X}$, respectively. $K_Y$ is the canonical divisor of the projective 4-space $Y$, and $k_{\overline{C}}$ that of $\overline{C}$.

4. The Euler number of the normalization of an algebraic threefold with ordinary singularities. By Theorem 3.1, the top polar classes $[M_3]$ of $\overline{X}$ is given by

$$[M_3] = (n - 1)^3h^3 - 3(n - 1)^2h^2 \cap s_2 - 3(n - 1)h \cap s_1 - s_0,$$

where $h$ denotes the hyperplane section class and $s_i$ $i$-th Segre class $s(\overline{J}, \overline{X})_i$ ($0 \leq i \leq 2$) and $n = \deg \overline{X}$, the degree of $\overline{X}$ in $Y$. We put

$$m = \deg \overline{D}, \ t = \deg \overline{T}, \ \gamma = \deg \overline{C} \text{ and } \#\Sigma \overline{C} = \text{the cardinal number of } \Sigma \overline{C}.$$ 

Then, by Proposition 3.6,

$$\begin{align*}
\left\{ \begin{array}{l}
\deg s_2 = 2m \\
\deg s_1 = -nm + 2\gamma - 3t \\
\deg s_0 = n^2m - 2m^2 + 5nt - 5\gamma - \#\Sigma \overline{C} - \deg k_{\overline{C}}.
\end{array} \right.
\end{align*}$$

Consequently, the class $c$ of $\overline{X}$ is given by

$$c = \deg [M_3] = (n - 1)^3\deg \overline{X} - 3(n - 1)^2\deg s_2 - 3(n - 1)\deg s_1 - \deg s_0$$

$$= (n - 1)^3n - (4n^2 - 9n - 2m + 6)m + (4n - 9)t - (6n - 11)\gamma + \#\Sigma \overline{C} + \deg k_{\overline{C}}.$$

By this formula together with Proposition 2.3, we have the following:

**Theorem 4.1** The Euler number $\chi(X)$ of the non-singular normalization $X$ of an algebraic threefold $\overline{X}$ with ordinary singularities in $P^4(C)$ which is free from quadruple points is given by

$$\chi(X) = -n(n^3 - 5n^2 + 10n - 10) + (4n^2 - 15n - 2m + 20)m - (4n - 15)t$$

$$+ (6n - 15)\gamma - \#\Sigma \overline{C} - \deg k_{\overline{C}}.$$ 

Here $n = \deg \overline{X}$, $m = \deg \overline{D}$, $t = \deg \overline{T}$ and $\gamma = \deg \overline{C}$ are the degrees of $\overline{X}$, the singular locus, the triple point locus and the cuspidal point locus, respectively. $\#\Sigma \overline{C}$ is the cardinal number of the stationary point locus $\Sigma \overline{C}$, and $\deg k_{\overline{C}}$ the degree of the canonical divisor of the cuspidal point locus $\overline{C}$.

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References


