The Existence of a Good Linear Pencil of Hyperplane Sections of an Algebraic Threefold with Ordinary Singularities *†

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Abstract

In [T-2] and [T-3] we have proved a numerical formula which gives the Euler number of the (non-singular) normalization Y of an algebraic threefold with ordinary singularities X in $P^4(\mathbf{C})$. To prove this formula we have used a 'good' linear pencil of hypreplane sections on X. In §1 of this article we outline the proof of the existence of such a 'good' linear pencil, in which we apply Artin's algebraization theorem to an isolated ordinary double point. In §2 we give an elementary proof of the algebraization theorem for an isolated ordinary double point.

1 The Euler number of the normalization of an algebraic threefold with ordinary singularities

We begin with giving the definition of an algebraic threefold with ordinary singularities.

Definition 1.1 *An irreducible hypersurface X in the complex projective 4-space $P^4(\mathbf{C})$ is called an algebraic threehold with ordinary singularities if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbf{C}^4$ at every point of X:*

$$\begin{cases} (i)\ w=0\ \big(\textit{simple point}\big) & (ii)\ zw=0\ \big(\textit{ordinary double point}\big) \\ (iii)\ yzw=0\ \big(\textit{ordinary triple point}\big) & (iv)\ xyzw=0\ \big(\textit{ordinary quadruple point}\big) \\ (v)\ xy^2-z^2=0\ \big(\textit{cuspidal point}\big) & (vi)\ w(xy^2-z^2)=0\ \big(\textit{stationary point}\big), \end{cases}$$

where (x,y,z,w) is the coordinate on $\mathbf{C}^4$.

These singularities arise if we project a non-singular threefold embedded in a sufficiently higher dimensional complex projective space to its four dimensional linear subspace by a *generic* linear projection ([R]). This fact can also be proved by use of the classification theory of miltigerms of *locally stable* holomorphic maps ([M-3], [T-1]). Indeed, in the threefold case, the pair of dimensions of the source and target manifolds belongs to the so-called *nice range*([M-2]). Hence the multi-germ of a *generic* linear projection at the inverse image of any point of X is *stable*, i.e., stable under small deformations ([M-4]). Throughout this article we fix the notation as follows:

X : an algebraic threefold with ordinary singularities in $P^4(\mathbf{C})$,

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Y : the (non-singular) normalization of X,

D : the double surface (singular locus) of X,

T : the triple curve of X, precisely speaking, the closure of triplel point locus of X, since we always consider T contains the quadruple points and stationary points. T is equal to the singular locus of D,

C : the cuspidal curve of X, precisely speaking, the closure of cuspidal point locus of X, since we always consider C contains the stationary points,

Σq : the quadruple point locus of X,

Σs : the stationary point locus of X,

k_C : the canonical divisor of the cuspidal curve of X.

In [T2], [T3] we have proved a numerical formula which gives the Euler number of the (non-singular) normalization of analgebraic threefold with ordinary singularities X in the complex projective 4-space $P^4(\mathbf{C})$.

Theorem 1.2 ([T2], [T3]) *The Euler number $\chi(Y)$ of the (non-singular) normalization Y of an algebraic threefold X with ordinary singularities in $P^4(\mathbf{C})$ is given by*

$$\begin{aligned}\chi(Y) \quad = \quad & -n(n^3-5n^2+10n-10)+(4n^2-15n-2m+20)m-(4n-15)t \\ & +(6n-15)\gamma-\#\Sigma\overline{s}-deg\,k_{\overline{C}}+4\#\Sigma\overline{q}.\end{aligned}$$

Here $n = deg\,X$, $m = deg\,D$, $t = deg\,T$ and $\gamma = deg\,C$ are the degrees of X, the duble surface D, the triple curve T and the cuspidal curve C of X, respectively. $\#\Sigma s$ is the cardinal number of the stationary point locus Σs, $\#\Sigma q$ the cardinal number of the quadruple point locus Σq and $deg\,k_C$ the degree of the canonical divisor of the cuspidal curve C.

To prove this theorem we use a 'good' linear pencil (Lefschetz pencil) of hyperplane sections of X. We now expalin what a 'good' linear pencil of hyperplane sections of X is. Let P_∞ be a 2-dimensional linear subspace of $P^4(\mathbf{C})$ such that $C_\infty := P_\infty \cap X$ is an irreducible curve with ordinary double points in $P_\infty \simeq P^2(\mathbf{C})$. Let P be a 1-dimensional linear subspace of $P^4(\mathbf{C})$ situated in twisted position with respect to P_∞, i.e., the linear subspace $L(P_\infty, P)$ generated by P_∞ and P is equal to $P^4(\mathbf{C})$. Let $\pi : X \setminus C_\infty \to P$ be the linear projection with center C_∞, i.e., $\pi(x) := H_x \cap P$ for $x \in X \setminus C_\infty$, where $H_x = L(x, P_\infty)$ is the hyperplane generated by x and P_∞. We put $X_\lambda := H_\lambda \cap X$ for $\lambda \in P$ and $\mathcal{L}_X := \bigcup_{\lambda\in P} X_\lambda$. Then $\mathcal{L}_X$ is a linear system on X with the base point locus $Bs(\mathcal{L}_X) = C_\infty$. Let $f : Y \to X$ be the normalization map and $\mathcal{L}_Y := \bigcup_{\lambda\in P} Y_\lambda$ the pull-back of $\mathcal{L}_X$ to Y. A 'good' linear pencil of hyperplane sections of X is the one whose existence is guaranteed by the following theorem.

Theorem 1.3 ([T2], Theorem 2.1) *If we take P_∞ sufficiently general, there exists a finite set $\{\lambda_1, \cdots, \lambda_c\}$ of points of P such that*

(i) *Y_λ is non-singular for λ with $\lambda \neq \lambda_i$ ($1 \leq i \leq c$), and*

(ii) *Y_{λ_i} is a surface with only one isolated ordinary double point which is contained in $Y \setminus f^{-1}(C_\infty)$ for any i with $1 \leq i \leq c$,*

where c is the class of X.

The outline of the proof of this theorem is as follows: We consider the *Gauss map*

$$\Phi : X --> P^4(C)^\vee$$

defined by

$$\Phi(p) = \Big[\frac{\partial F}{\partial x_0}(p) : \frac{\partial F}{\partial x_1}(p) : \frac{\partial F}{\partial x_2}(p) : \frac{\partial F}{\partial x_3}(p) : \frac{\partial F}{\partial x_4}(p)\Big] \tag{1.1}$$

for $p \in X$, where F is the homogeneous polynomial defining X in $P^4(C)$, $[x_0 : x_1 : x_2 : x_3 : x_4]$ the homogeneous coordinate on $P^4(\mathbf{C})$, and $P^4(\mathbf{C})^\vee$ the dual projective space of $P^4(\mathbf{C})$. Φ is a rational map, which is not defined on the singular locus D of X. Let X^* be the closure in $X \times P^4(\mathbf{C})^\vee$ of the graph of Φ. We denote by $\pi_1 : X^* \to X$ the morphism induced by the projection to the first factor, and $\pi_2 : X^* \to P^4(\mathbf{C})^\vee$ the one induced by the projection to the second factor. We call $\pi_1 : X^* \to X$ the *Nash blow-up* of X. Note that the rational map Φ can be extended to X^* and X^* is minimal among the varieties with such property. In our case, since X is a hypersurface, X^* coincides with the the blow-up of the Jacobian ideal of X ([N], Remark 2, p.300). We denote by $X^\vee$ the image of X^* by $\pi_2 : X^* \to P^4(\mathbf{C})^\vee$, and call it the *dual variety* of X. The dimension of $X^\vee$ is not less than 1, nor greater than 3 ([H], Example 15.22., p.196).

We are now going to define an algebraic subset B in $P^4(\mathbf{C})^\vee$, whose points correspond to hyperplanes in $P^4(\mathbf{C})$ being in *bad* positions in some sense at their intersecting points with the cuspidal curve C, or stationary point locus Σs of X. Let p be a point of C, or Σs. Then there is an open neighborhood U of p and a complex analytic local coordinats (x, y, z, w) with center p such that the defining equation of X is given by one of the following:

$$xy^2 - z^2 = 0 \tag{1.2}$$

$$w(xy^2 - z^2) = 0. \tag{1.3}$$

Let $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ be a linear affine coordinate with center p, and H a hyperplane passing through p, defined by the equation

$$\sum_{i=1}^{i=4} a_i \zeta_i = 0 \quad (a_i \in \mathbf{C},\ 1 \le i \le 4). \tag{1.4}$$

We say H is in a *bad* position at the point p, if the coefficients of the equation (1.4) satisfy the following two conditions:

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) = 0, \tag{1.5}$$

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) = 0. \tag{1.6}$$

We define B_p to be the algebraic subset of $P^4(C)^\vee$ consisting of all points which corresponds to hyperplanes in $P^4(C)$ passing through p and being in a bad position at p in the sense defined above. We define an algebraic subset B of $P^4(C)^\vee$ by

$$B := \textstyle\bigcup_{p \in C} B_p \tag{1.7}$$

Here we should note that the stationary points are included in C, and since $\dim B_p = 1$, the codimension of B is greater than 1. We choose a line L^* in $P^4(\mathbf{C})^\vee$ which satisfies all of the following conditions:

$$L^* \cap \{X^\vee \setminus \Phi(X_{sm})\} = \emptyset, \tag{1.8}$$

$$L^* \cap (X^\vee)_{sing} = \emptyset, \tag{1.9}$$

$$L^* \cap B = \emptyset, \tag{1.10}$$

$$L^* \text{ intersects transversely with } \Phi(X_{sm}) \setminus (X^\vee)_{sing}, \tag{1.11}$$

where X_{sm} denotes $X \setminus D$, the simple point locus of X, and $(X^\vee)_{sing}$ the singular point locus of $X^\vee$. This is always possible because all the codimensions of $X^\vee \setminus \Phi(X_{sm})$, $(X^\vee)_{sing}$ and B are greater than 1 in $P^4(\mathbf{C})^\vee$. Note that the cardinal number of the set $L^* \cap \{\Phi(X_{sm}) \setminus (X^\vee)_{sing}\}$ is nothing but the *class* of X. We denote by H_λ the hyperplane in $P^4(\mathbf{C})$ corresponding to each

$\lambda \in L^*$. We put $X_\lambda := X \cap H_\lambda$ and consider the linear pencil

$$\mathcal{L}_X = \bigcup_{\lambda \in L^*} X_\lambda$$

of hyperplane sections of X. Since the line L^* is choosen so that the conditions from (1.8) through (1.11) are satisfied, we can show that the assertions (i) and (ii) of the theorem holds for the linear pencil $\mathcal{L}_X = \bigcup_{\lambda \in L^*} X_\lambda$. Here we sketch the proof of the assertion (ii). By the manner of choosing the line L^* in $P^4(\mathbf{C})^\vee$, the hyperplane H_{λ_i} is tangent to X at only one point, say q, of X_{sm}. Therefore X_{λ_i} is non-singular at all but one point q of $X_{\lambda_i} \cap X_{sm}$. To prove that X_{λ_i} has an isolated ordinary double point at q, we assume that the homogeneous coordinate $[x_0 : x_1 : x_2 : x_3 : x_4]$ of q is $[1 : 0 : 0 : 0 : 0]$ and H_{λ_i} is defined by $x_4 = 0$. We put $\zeta_i = x_i/x_0$ $(1 \le i \le 4)$, and use this linear affine coordinate $(\zeta_1, \cdots, \zeta_4)$ in the subsequent arguments. Then X is defined by $F(1, \zeta_1, \zeta_2, \zeta_3, \zeta_4) = 0$, q is the origin $(0, \cdots, 0)$, and H_{λ_i} is defined by $\zeta_4 = 0$. Since the tangent hyperplane to X at q is the hyperplane $H_{\lambda_i} : \zeta_4 = 0$, we have

$$\frac{\partial F}{\partial \zeta_i}(1, 0, \cdots, 0) = 0 \qquad (1 \le i \le 3) \tag{1.12}$$

$$\frac{\partial F}{\partial \zeta_4}(1, 0, \cdots, 0) \neq 0. \tag{1.13}$$

Because of (1.13), there is an analytic function $\phi(\zeta_1, \zeta_2, \zeta_3)$ of the variables $\zeta_1, \zeta_2, \zeta_3$ defined in a neighborhood of the origin, which satisfies the following:

$$\phi(0, 0, 0) = 0, \tag{1.14}$$

$$F(1,\, \zeta_1,\, \zeta_2,\, \zeta_3,\, \phi(\zeta_1, \zeta_2, \zeta_3)) \equiv 0 \quad \text{(locally)}. \tag{1.15}$$

This means that the defining equation of X in a neighborhood of q is given by

$$\zeta_4 = \phi(\zeta_1, \zeta_2, \zeta_3). \tag{1.16}$$

By the same reasoning as before, we have

$$\frac{\partial \phi}{\partial \zeta_i}(0, 0, 0) = 0 \qquad (1 \le i \le 3) \tag{1.17}$$

Hence ϕ is expresed as

$$\phi = \sum_{1 \le i, j \le 3} \frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j}(0)\, \zeta_i \zeta_j + O(3). \tag{1.18}$$

If we regard $(\zeta_1, \zeta_2, \zeta_3)$ as a local coordinate on H_{λ_i}, X_{λ_i} is defined by $\phi(\zeta_1, \zeta_2, \zeta_3) = 0$ in H_{λ_i}. Therefore, if we prove

$$\det\Big(\frac{\partial^2 \phi}{\partial \zeta_i \partial \zeta_j}(0)\Big) \neq 0 \tag{1.19}$$

then we can conclude that, after suitable change of local coordinates, the defining equation of X_{λ_i} will become

$$\phi = \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + O(|\zeta|^3)$$

in a neighborhood of the origin in H_{λ_i}. (1.19) can be proved by evaluating the Hessian $\det(\partial^2 F / \partial x_i \partial x_j)$ of the homogeneous polynomial F at $q = [1 : 0 : 0 : 0]$. Then, by Artin's algebraization thorem ([A], Thorem 3.8), we conclude that X_{λ_i} is analytically equivalent to an isolated ordinary double point defined by

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 0$$

This completes the prof of the assertion (ii) of the theorem.

Let $\sigma : \widehat{Y} \to Y$ be the blowing-up along $f^{-1}(C_\infty)$. Then the proper inverse image $\mathcal{L}_{\widehat{Y}} := \bigcup_{\lambda\in P} \widehat{Y_\lambda}$ of $\mathcal{L}_Y := \bigcup_{\lambda\in P} Y_\lambda$ by σ gives a fibering of $\widehat{Y}$ over $P \simeq P^1(\mathbf{C})$ with finite singular fibers $\widehat{Y}_{\lambda_i}$ $(1\le i\le c)$. Therefore, the calculation of the Euler number of $\widehat{Y}$, hence that of Y, is reduced to the calculation of the *class* c of X. This can be done by the calculation of the *Segre classes* $s(J,X)_i$ $(0\le i\le 2)$ of J in X owing to Piene's Pücker formula for hypersurfaces ([R]), where J is the singular subscheme of X defined by the Jacobian ideal of X.

2 Algebraization of an isolated ordinary double point (an elementary proof)

In this section we shall give an elementary proof of the algebraization theorem for an isolated ordinary double point.

Theorem 2.1 *Let ϕ be a convergent power series of the form*

$$\phi = \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + O(|\zeta|^3).$$

Then there exist convergent power serieses $f(\zeta)$, $g(\zeta)$, $h(\zeta)$ such that

$$\text{(2.1)} \qquad \begin{cases} f(\zeta) = \zeta_1 + O(|\zeta|^2), \\ g(\zeta) = \zeta_2 + O(|\zeta|^2), \\ h(\zeta) = \zeta_3 + O(|\zeta|^2), \quad \textit{and} \end{cases}$$

$$\text{(2.2)} \qquad \phi = f(\zeta)^2 + g(\zeta)^2 + h(\zeta)^2.$$

Proof: We put

$$F(\zeta_1,\zeta_2,\zeta_3) = \zeta_1^2 + \zeta_2^2 + \zeta_3^2$$

We will construct convergent power serieses $f(\zeta), g(\zeta), h(\zeta)$ which are of the forms (2.1), and satisfy

$$\text{(2.3)} \qquad \phi = F(f,g,h)$$

In what follows we shall express a power series $P(\zeta)$ in $(\zeta_1,\zeta_2,\zeta_3)$ as

$$P(\zeta) = \sum_{k\ge 0} P_k(\zeta)$$

where $P_k(\zeta)$ is a homogeneous polynomial in $(\zeta_1,\zeta_2,\zeta_3)$ of degree k. With this notation we define

$$P^\mu(\zeta) := \sum_{0\le k\le \mu} P_k(\zeta)$$

for a power series $P(\zeta)$.

I) Existence of formal solutions:

First we prove the existence of formal solutions of the equation (2.3). In what follows, for any power serieses $P(\zeta), Q(\zeta)$ in $(\zeta_1, \zeta_2, \zeta_3)$, we indicate by writing

$$P(\zeta) \equiv_\mu Q(\zeta)$$

that the power series $P(\zeta) - Q(\zeta)$ in $(\zeta_1, \zeta_2, \zeta_3)$ contains no term of degree$\leq \mu$. With this notation, the equation (2.3) is equivalent to the equations of congruences

$$(2.4)_\mu \qquad F(f^{\mu-1} + f_\mu,\ g^{\mu-1} + g_\mu,\ h^{\mu-1} + h_\mu) \equiv_{\mu+1} \phi^\mu + \phi_{\mu+1} \qquad (\mu \geq 1).$$

First, we put

$$(2.5) \qquad \begin{cases} f_0(\zeta) = g_0(\zeta) = h_0(\zeta) = 0, \quad \textit{and} \\ \\ f_1(\zeta) = \zeta_1,\ \ g_1(\zeta) = \zeta_2,\ \ h_1(\zeta) = \zeta_3. \end{cases}$$

We construct $f^\mu(\zeta), g^\mu(\zeta), h^\mu(\zeta)$ satisfying $(2.4)_\mu$ by induction on μ. For $\mu = 1$, we define $f^1(\zeta) = f_0(\zeta) + f_1(\zeta)$, $g^1(\zeta) = g_0(\zeta) + g_1(\zeta)$, $h^1(\zeta) = h_0(\zeta) + h_1(\zeta)$ by (2.5). Then $(2.4)_\mu$ holds, because $\phi^1 \equiv 0$ and $\phi_2 = \zeta_1^2 + \zeta_2^2 + \zeta_3^2$. Next we suppose that $f^{\mu-1}(\zeta)$, $g^{\mu-1}(\zeta)$, $h^{\mu-1}(\zeta)$ satisfying $(2.4)_{\mu-1}$ are already determined. Since

$$\begin{aligned} &F(f^{\mu-1} + f_\mu,\ g^{\mu-1} + g_\mu,\ h^{\mu-1} + h_\mu) \\ &= (f^{\mu-1} + f_\mu)^2 + (g^{\mu-1} + g_\mu)^2 + (h^{\mu-1} + h_\mu)^2 \\ &\equiv_{\mu+1} (f^{\mu-1})^2 + (g^{\mu-1})^2 + (h^{\mu-1})^2 + 2(\zeta_1 f_\mu + \zeta_2 g_\mu + \zeta_3 h_\mu) + (f_\mu)^2 + (g_\mu)^2 + (h_\mu)^2 \end{aligned}$$

and, since $2\mu \geq \mu + 2$, we have

$$(f_\mu)^2 + (g_\mu)^2 + (h_\mu)^2 \equiv_{\mu+1} 0.$$

Hence $(2.4)_\mu$ is equivalent to

$$F(f^{\mu-1}, g^{\mu-1}, h^{\mu-1}) + 2(\zeta_1 f_\mu + \zeta_2 g_\mu + \zeta_3 h_\mu) \equiv_{\mu+1} \phi^\mu + \phi_{\mu+1}.$$

Therefore, if we define a homogeneous polynomial $\Gamma_{\mu+1}(\zeta_1, \zeta_2, \zeta_3)$ in $(\zeta_1, \zeta_2, \zeta_3)$ of degree $\mu + 1$ by the equation of congruence

$$(2.6)_\mu \qquad \Gamma_{\mu+1}(\zeta_1, \zeta_2, \zeta_3) \equiv_{\mu+1} \frac{1}{2}\{\phi^{\mu+1} - F(f^{\mu-1}, g^{\mu-1}, h^{\mu-1})\},$$

then the equation $(2.4)_\mu$ is reduced to

$$(2.7)_\mu. \qquad \zeta_1 f_\mu + \zeta_2 g_\mu + \zeta_3 h_\mu \equiv_{\mu+1} \Gamma_{\mu+1}(\zeta_1, \zeta_2, \zeta_3)$$

Obviously, there exist homogeneous polynomials f_μ, g_μ, h_μ satisfying the equation above.

II) Convergence of formal solutions:

We put

$$(2.8) \qquad A(\zeta_1, \zeta_2, \zeta_3) = \frac{b}{16c} \sum_{\nu=1}^{\infty} \frac{c^\nu}{\nu^2} (\zeta_1 + \zeta_2 + \zeta_3)^\nu.$$

Since

$$\lim_{\nu \to \infty} \frac{\left| \dfrac{c^{\nu+1}}{(\nu+1)^2} \right|}{\left| \dfrac{c^\nu}{\nu^2} \right|} = \lim_{\nu \to \infty} \left| \left(\frac{\nu}{\nu+1} \right)^2 c \right| = |c|,$$

the series $A(\zeta_1, \zeta_2, \zeta_3)$ converges in the domain $|\zeta_1 + \zeta_2 + \zeta_3| < 1/|c|$. We note that the following holds for $A(\zeta)$:

$$A(\zeta)^\nu << \left(\frac{b}{c}\right)^{\nu-1} A(\zeta) \qquad (\nu \geq 2). \tag{2.9}$$

Here and in what follows, for two power serieses

$$\begin{aligned} a(\zeta) &= \sum_{\nu_1\geq 0, \nu_2 \geq 0, \nu_3 \geq 0} a_{\nu_1\nu_2\nu_3} \zeta_1^{\nu_1} \zeta_2^{\nu_2} \zeta_3^{\nu_3}, \quad \textit{and} \\ b(\zeta) &= \sum_{\nu_1\geq 0, \nu_2 \geq 0, \nu_3 \geq 0} b_{\nu_1\nu_2\nu_3} \zeta_1^{\nu_1} \zeta_2^{nu_2} \zeta_3^{\nu_3}, \end{aligned}$$

we indicate by writing

$$a(\zeta) << b(\zeta)$$

that

$$|a_{\nu_1\nu_2\nu_3}| \leq |b_{\nu_1\nu_2\nu_3}|$$

holds for every multi-index (ν_1, ν_2, ν_3). By induction on μ, we are now going to prove that

$$(2.10)_\mu \qquad f^\mu(\zeta) - \zeta_1, \quad g^\mu(\zeta) - \zeta_2, \quad h^\mu(\zeta) - \zeta_3 << A(\zeta_1, \zeta_2, \zeta_3) \qquad (\mu \geq 1)$$

hold if the constant b, c are choosen properly. For $\mu = 1$, since $f^1(\zeta) = \zeta_1$, $g^1(\zeta) = \zeta_2$, $h^1(\zeta) = \zeta_3$, there is nothing to prove. Next we suppose that $f^{\mu-1}(\zeta)$, $g^{\mu-1}(\zeta)$, $h^{\mu-1}(\zeta)$ satisfying $(2.10)_\mu$ are already determined. Recall that f_μ, g_μ, h_μ are determined so that $(2.7)_\mu$ holds. To estimate $\Gamma_{\mu+1}(\zeta_1, \zeta_2, \zeta_3)$ by a superior power series in $(\zeta_1, \zeta_2, \zeta_3)$, we set

$$A_0(\zeta_1, \zeta_2, \zeta_3, t_1, t_2, t_3) = \frac{b_0}{c_0} \sum_{\nu \geq 1} c_0^\nu (\zeta_1 + \zeta_2 + \zeta_3 + t_1 + t_2 + t_3)^\nu,$$

where b_0, c_0 are non-negative real numbers. Taking the constant b_0, c_0 properly, we may assume that

$$\begin{aligned} &\frac{1}{2}\{\phi(\zeta_1, \zeta_2, \zeta_3) - F(\zeta_1 + t_1, \zeta_2 + t_2, \zeta_3 + t_3)\} \\ &\qquad << \frac{b_0}{c_0} \sum_{\mu \geq 1} c_0^\mu (\zeta_1 + \zeta_2 + \zeta_3 + t_1 + t_2 + t_3)^\nu. \end{aligned} \tag{2.11}$$

Subdtituting $f^{\mu-1} - \zeta_1$, $g^{\mu-1} - \zeta_2$, $h^{\mu-1} - \zeta_3$ for t_1, t_2, t_3 in (2.11), respectively, we have

$$\begin{aligned} &\frac{1}{2}\{\phi(\zeta_1, \zeta_2, \zeta_3) - F(f^{\mu-1}, g^{\mu-1}, h^{\mu-1})\} \\ &<< \frac{b_0}{c_0} \sum_{\mu \geq 2} c_0^\mu \{\zeta_1 + \zeta_2 + \zeta_3 + (f^{\mu-1} - \zeta_1) + (g^{\mu-1} - \zeta_2) + (h^{\mu-1} - \zeta_3)\}^\nu. \end{aligned} \tag{2.12}$$

If we take the constant b in (2.8) sufficiently large so that $b/16 > 1$, we have

$$\zeta_1, \ \ \zeta_2, \ \ \zeta_3 << A(\zeta_1, \zeta_2, \zeta_3).$$

Furthermore, by the induction hypothesis,

$$f^{\mu-1} - \zeta_1, \quad g^{\mu-1} - \zeta_2, \quad h^{\mu-1} - \zeta_3 << A(\zeta_1, \zeta_2, \zeta_3).$$

Hence, using (2.9),

$$\{\zeta_1+\zeta_2+\zeta_3+(f^{\mu-1}-\zeta_1)+(g^{\mu-1}-\zeta_2)+(h^{\mu-1}-\zeta_3)\}^\nu$$
$$<< \{6A(\zeta_1,\zeta_2,\zeta_3)\}^\nu = 6^\nu A(\zeta_1,\zeta_2,\zeta_3)^\nu << 6^\nu\left(\frac{b}{c}\right)^{\nu-1} A(\zeta_1,\zeta_2,\zeta_3)$$

Therefore, by (2.12),

$$\begin{aligned}\frac{1}{2}\{\phi(\zeta_1,\zeta_2,\zeta_3)-F(f^{\mu-1},g^{\mu-1},h^{\mu-1})\} &<< \frac{b_0}{c_0}\sum_{\nu=2}^{\infty} c_0^\nu 6^\nu\left(\frac{b}{c}\right)^{\nu-1} A(\zeta_1,\zeta_2,\zeta_3)\\ &= \frac{b_0c}{c_0b}\{\sum_{\nu=2}^{\infty}\left(\frac{6c_0b}{c}\right)^\nu\}A(\zeta_1,\zeta_2,\zeta_3).\end{aligned} \tag{2.13}$$

If we take the constant c so large that $6c_0b/c < 1/2$, we have

$$\sum_{\nu=2}^{\infty}\left(\frac{6c_0b}{c}\right)^\nu = \left(\frac{6c_0b}{c}\right)^2\sum_{\nu=0}^{\infty}\left(\frac{6c_0b}{c}\right)^\nu = \left(\frac{6c_0b}{c}\right)^2\frac{1}{1-\left(\frac{6c_0b}{c}\right)} < 2\left(\frac{6c_0b}{c}\right)^2.$$

Hence, by (2.13),

$$\frac{1}{2}\{\phi(\zeta_1,\zeta_2,\zeta_3)-F(f^{\mu-1},g^{\mu-1},h^{\mu-1})\} << \frac{72bb_0c_0}{c}A(\zeta_1,\zeta_2,\zeta_3).$$

Since

$$\Gamma_{\mu+1}(\zeta_1,\zeta_2,\zeta_3) \equiv_{\mu+1} \frac{1}{2}\{\phi(\zeta_1,\zeta_2,\zeta_3)-F(f^{\mu-1},g^{\mu-1},h^{\mu-1})\},$$

if we take the constant c so large that $72bb_0c_0/c < 1$,

$$\Gamma_{\mu+1}(\zeta_1,\zeta_2,\zeta_3) << A(\zeta_1,\zeta_2,\zeta_3).$$

Therefore, since $\zeta_1 f_\mu + \zeta_2 g_\mu + \zeta_3 h_\mu = \Gamma_{\mu+1}(\zeta_1,\zeta_2,\zeta_3)$,

$$f_\mu, \quad g_\mu, \quad h_\mu << A(\zeta_1,\zeta_2,\zeta_3).$$

Consequently,

$$f^\mu-\zeta_1, \quad g^\mu-\zeta_2, \quad h^\mu-\zeta_3 << A(\zeta_1,\zeta_2,\zeta_3)$$

as desired. Since the series $A(\zeta_1,\zeta_2,\zeta_3)$ converges in the domain $|\zeta_1+\zeta_2+\zeta_3| < 1/|c|$, $f(\zeta)$, $g(\zeta)$, $h(\zeta)$ converge in the same domain. This completes the proof of the theorem. ∎

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