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INFINITESIMAL MIXED TORELLI PROBLEM FOR ALGEBRAIC SURFACES WITH ORDINARY SINGULARITIES

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Abstract

Thanks to Deligne's result ([1]) there exists a *mixed Hodge structure* on the cohomology of a *singular* complex projective variety. Hence we may consider *infinitesimal mixed Torelli problem* for singular complex projective varieties like infinitesimal Torelli problem for non-singular complex projective ones. But, for this purpose, we need to consider an *equisingular* family of singular complex projective varieties in some sense so that from such a family there arises naturally a *variation of mixed Hodge structures*. In this talk we consider a *locally trivial* deformation family of complex projective *surfaces with ordinary singularities*. Note that every non-singular compact complex algebraic surface has its birationally equivalent model of a hypersurface with ordinary singularities in $\mathbb{P}^3(\mathbb{C})$. For algebraic surfaces with ordinary singularities the mixed Hodge structure on its cohomology can be described by use of its *cubic hyper-resolution* in the sense of F. Guillén, V. Navarro Aznar et al. ([4]). Using this description, we can formulate the infinitesimal mixed Torelli problem cohomologically and give sufficient conditions for the infinitesimal mixed Torelli problem to be affirmatively solved for complex projective surfaces with ordinary singularities (Theorem 3.2).

§0 Hodge structure, mixed Hodge structure and variation of mixed Hodge structures

For the reader's convenience we begin by recalling the definitions of *Hodge structure*, *mixed Hodge structure* and *variation of mixed Hodge structures*. Let A be $\mathbb{Z}$ or $\mathbb{Q}$. For an A-module H_A, the complex conjugation can be defined on $H_{\mathbb{C}} := \mathbb{C} \otimes_A H_A$. A filtration $F = \{F^p\}$ of $H_{\mathbb{C}}$ by $\mathbb{C}$-vector subspaces admits its conjugate filtration $\overline{F}$ such that $(\overline{F})^p = \overline{F^p}$.

0.1 Definition. An A–Hodge structure of weight ℓ consists of

(i) an A–module of finite type H_A, and

(ii) a finite decreasing filtration $F = \{F^p\}$ of $H_{\mathbb{C}}$ by $\mathbb{C}$-vector subspaces (Hodge filtration) such that $Gr_F^p Gr_{\overline{F}}^q(H_{\mathbb{C}}) = 0$ for $p + q \neq \ell$.

Note that the relation above implies that the subspaces

$$H^{p,q} := F^p \cap \overline{F^q}$$

give a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=\ell} H^{p,q}, \qquad H^{p,q} = \overline{H^{q,p}}.$$

0.2 Definition. A mixed Hodge structure consists of

(i) a free $\mathbb{Z}$-module of finite type $H_{\mathbb{Z}}$,

(ii) a finite increasing filtration $W = \{W_q\}$ of $H_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ by $\mathbb{Q}$-vector subspaces (weight filtration), and

(iii) a finite decreasing filtration $F = \{F^p\}$ of $H_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} H_{\mathbb{Z}}$ by $\mathbb{C}$-vector subspaces (Hodge filtration), satisfying the condition that

$$Gr_q^W(H) := (Gr_q^W(H_{\mathbb{Q}}), Gr_q^W(H_{\mathbb{C}}), F)$$

forms a $\mathbb{Q}$-Hodge structure of weight q for every q.

0.3 Definition. A variation of mixed Hodge structures on a complex manifold M consists of

(i) a local system $V^{\mathbb{Z}}$ of free $\mathbb{Z}$-module of finite type on M,

(ii) a finite increasing filtration $\mathbb{W} = \{W_q\}$ of $V^{\mathbb{Q}} := V^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by sub-local systems of $\mathbb{Q}$-vector spaces, and

(iii) a finite decreasing filtration $\mathbb{F} = \{\mathcal{F}^p\}$ of $\mathcal{V} := V^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_M$ ($\mathcal{O}_M$: the structure sheaf of M) by holomorphic subbundles, satisfying

(a)(the Griffiths transversality)

$\nabla \mathcal{F}^p \subset \Omega_M^1 \otimes \mathcal{F}^{p-1}$,

where ∇ denotes the Gauss-Manin connection on $\mathcal{V}$, and

(b) $(V^{\mathbb{Z}}, \mathbb{W}, \mathbb{F})$ defines a mixed Hodge structuer at each point $t \in M$.

§1 Algebraic surfaces with ordinary singularities and their cubic hyper-resolution

1.1 Definition. A 2-dimensional complex projective variety S is said to be an algebraic surface with *ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurfaces at the origin of the complex 3-space $\mathbb{C}^3$ at every point of S :

$$\begin{cases} (i)\ z = 0\ \text{(simple point)} & (ii)\ yz = 0\ \text{(ordinary double point)} \\ (iii)\ xyz = 0\ \text{(ordinary triple point)} & (iv)\ xy^2 - z^2 = 0\ \text{(cuspidal point)} \end{cases}$$

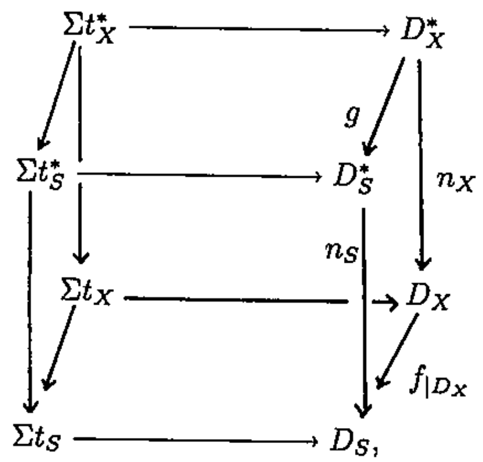
We denote by D_S the singular locus of S, and call it the *double curve* of S. D_S is a singular curve with triple points. We denote by Σt_S the triple point locus of

S, and by Σc_S the cuspidal point locus of S. Let $f : X \to S$ be the normalization. We put $D_X := f^{-1}(D_S)$ and $\Sigma t_X := f^{-1}(\Sigma t_S)$. D_X is a singular curve with nodes and Σt_X coincides with the set of nodes of D_X. Then a *2-resolution* of S in the sense of F. Guillén, V. Navarro Aznar et al. ([4]) is obtained as follows:

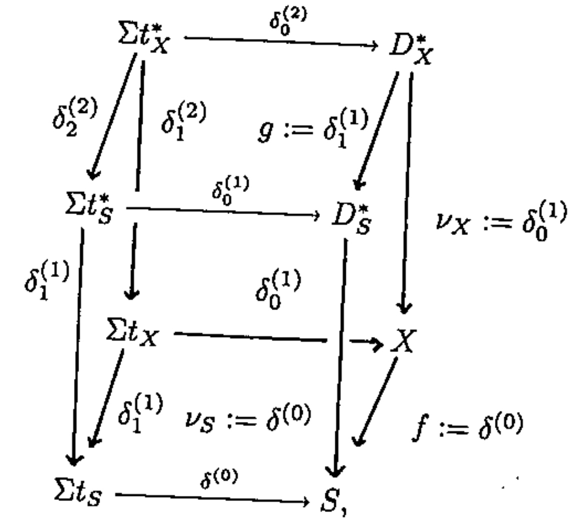
$$\begin{array}{ccc} D_X & \longrightarrow & X \\ {\scriptstyle f_{|D_X}}\downarrow & & \downarrow{\scriptstyle f} \\ D_S & \longrightarrow & S, \end{array} \tag{1.1}$$

where $f_{|D_X}$ denotes the restriction of f to D_X, and horizontal arrows are inclusion maps. We consider the map $f_{|D_X} : D_X \to D_S$ as a *0-cubic complex projective variety* (cf. [4]). A 2-resolution of the 0-cubic complex projective variety $f_{|D_X} : D_X \to D_S$ in the sense of F. Guillén, V. Navarro Aznar et al. is obtained as follows:

(1.2)



where $n_S : D_S^* \to D_S$ and $n_X : D_X^* \to D_X$ are the normalizations, $g : D_X^* \to D_S^*$ the lifting of the map $f_{|D_X} : D_X \to D_S$, and $\Sigma t_S^* := n_S^{-1}(\Sigma t_S)$, $\Sigma t_X^* := n_X^{-1}(\Sigma t_X)$. Replacing $f_{|D_X} : D_X \to D_S$ in (1.2) by $f : X \to S$ in (1.1), we obtain the following *cubic hyper-resolution* of S in the sense of F. Guillén, V. Navarro Aznar et al.:



where ν_S and ν_X are the composites of the normalizations and the inclusion maps. We put $X_0 := X \coprod D_S^* \coprod \Sigma t_S$ (disjoint union), $X_1 := D_X^* \coprod \Sigma t_X, \coprod \Sigma t_S^*$, $X_2 :=$ Σt_X^*, $\pi_2 := \delta^{(0)} \circ \delta_{i_1}^{(1)} \circ \delta_{i_2}^{(2)} : X_2 \to S$, $\pi_1 := \delta^{(0)} \circ \delta_{i_1}^{(1)} : X_1 \to S$, and $\pi_0 := \delta^{(0)} :$ $X_0 \to S$, where $i_1 = 0, 1$ and $i_2 = 0, 1, 2$. Then the *semi-simplicial hyper-resolution* of S associated to this cubic hyper-resolution is as follows (cf. [3]):

$$X_2 \;\overset{\delta_0^{(2)}}{\underset{\delta_2^{(2)}}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\delta_1^{(2)}}}}}}\; X_1 \;\overset{\delta_0^{(1)}}{\underset{\delta_1^{(1)}}{\rightrightarrows}}\; X_0 \xrightarrow{\delta^{(0)}} S$$

We denote symbolically this semi-simplicial hyper-resolution of S by $\pi. : X. \to S$. We denote by $D^+(S, \mathbb{Z})$ the derived category of lower bounded complexes of sheaves of $\mathbb{Z}$-modules over S. We define $K \in Ob(D^+(S, \mathbb{Z}))$ by

$$K: \quad 0 \to \pi_{0*}\mathbb{Z}_{X_0} \xrightarrow{d^0} \pi_{1*}\mathbb{Z}_{X_1} \xrightarrow{d^1} \pi_{2*}\mathbb{Z}_{X_2} \to 0 \quad (K^0 = \pi_{0*}\mathbb{Z}_{X_0}),$$

where $d^0 := \delta_0^{(1)*} - \delta_1^{(1)*}$ and $d^1 := \delta_0^{(2)*} - \delta_1^{(2)*} + \delta_2^{(2)*}$. Then $K = \mathbb{Z}_S$ in $D^+(S, \mathbb{Z})$. We define a so-called *weight filtration* W on $K_{\mathbb{Q}} := K \otimes \mathbb{Q} \in Ob(D^+(S, \mathbb{Q}))$ by $W_{-q}(K_{\mathbb{Q}}) := \sigma._{\geq q}\pi._*\mathbb{Q}_{X.}$ (stupid filtration). Then $(K_{\mathbb{Q}}, W) \in Ob(D^+F(S, \mathbb{Q}))$, where $D^+F(S, \mathbb{Q})$ denotes the derived category of filtered, lower bounded complexes of sheaves of $\mathbb{Q}$-modules over S. By calculation we can prove that $K_{\mathbb{C}} :=$ $K \otimes \mathbb{C}$ is quasi-isomorphic to $s(\pi._*\Omega^{\cdot}_{X.})$, where $\Omega^{\cdot}_{X_i}$ (i=0,1,2) denotes the holomorphic de Rham complex over X_i and $s(\pi._*\Omega^{\cdot}_{X.})$ the simple complex associated to the double complex $\pi._*\Omega^{\cdot}_{X.}$. We define a so-called *Hodge filtration* F on $K_{\mathbb{C}} \simeq s(\pi._*\Omega^{\cdot}_{X.})$ by $F^p(s(\pi._*\Omega^{\cdot}_{X.})) := s(\sigma_{q \geq p}\pi._*\Omega^q_{X.})$. Then the data: $\mathbb{Z}_S$, $(\pi._*\mathbb{Q}_{X.}, W)$, $\mathbb{Q}_S \simeq \pi._*\mathbb{Q}_{X.}$, $(s(\pi._*\Omega^{\cdot}_{X.}), W, F)$, $(\pi._*\mathbb{Q}_{X.}, W) \otimes \mathbb{C} \simeq (s(\pi._*\Omega^{\cdot}_{X.}), W)$, is a *cohomological mixed Hodge complex* in the sense of Deligne. Hence the filtration $W[\ell]$ $(W[\ell]_q := W_{q-\ell}$, the shift of the filtration degree to the right by $\ell)$ on $H^\ell(S, \mathbb{Q}) \simeq \mathbb{H}^\ell(X., \mathbb{Q}_{X.}) \simeq H^\ell(\mathbb{R}\Gamma(S, s(\pi._*\mathbb{Q}_{X.})))$ and the filtration F on

$H^\ell(S,\mathbb{C}) \simeq \mathbb{H}^\ell(X_\cdot,\mathbb{C}_{X_\cdot}) \simeq \mathbb{H}^\ell(X_\cdot,\Omega^\cdot_{X_\cdot}) \simeq H^\ell(\mathbb{R}\Gamma(S,s(\pi_{\cdot *}\Omega^\cdot_{X_\cdot})))$ $(0 \le \ell \le 4)$ define a mixed Hodge structure on $H^\ell(S,\mathbb{Z})$ (modulo torsion). Since the spectral sequence with respect to the weight fltration $W[\ell]$ abuting to $H^\ell(S,\mathbb{Q})$ degenerates at E_2-term (cf. [1], [2]), we have:

1.2 Proposition. *For every pair of integers (ℓ,k) with $0 \le \ell \le 4$ and $\ell-2 \le k \le \ell$,*

$$Gr_k^{W[\ell]}H^\ell(S,\mathbb{Q}) \simeq \frac{Ker\{H^k(X_{\ell-k},\mathbb{Q}) \xrightarrow{d^{\ell-k}} H^k(X_{\ell-k+1},\mathbb{Q})\}}{Im\{H^k(X_{\ell-k-1},\mathbb{Q}) \xrightarrow{d^{\ell-k-1}} H^k(X_{\ell-k},\mathbb{Q})\}}.$$

§2 Variation of mixed Hodge structures arising from a locally trivial family of algebraic surfaces with ordinary singularities

Now we consider a *locally trivial* analytic family of algebraic surfaces with ordinary singularities, from which there arises naturally a variation of mixed Hodge structures.

2.1 Definition. A *locally trivial* analytic family of algebraic surfaces with ordinary singularities, parametrized by a complex manifold M, is defined to be a triplet $(\mathfrak{S},\pi,M)$ such that;

(i) $\pi : \mathfrak{S} \longrightarrow M$ is a surjective holomorphic map from a complex space to a complex manifold,

(ii) $S_t := \pi^{-1}(t)$ is an algebraic surface with ordinary singularities for every point $t \in M$,

(iii) for every point $p \in \mathfrak{S}$, there exist open neighborhoods $\mathcal{U}$ of p in $\mathfrak{S}$, V of $\pi(p)$ in M with $\pi(\mathcal{U}) = V$, and a biholomorphic map $\phi : \mathcal{U} \longrightarrow U \times V$, where we define $U := \mathcal{U} \cap S_{\pi(p)}$, such that;

(a) the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\phi} & U \times V \\ & {\scriptstyle \pi|_\mathcal{U}} \searrow \quad \swarrow {\scriptstyle Pr_V} & \\ & V & \end{array}$$

commuts,

(b) $\phi_{|U\times p} := id_{U\times p}$.

We denote by $\mathfrak{X}_\cdot \xrightarrow{\varpi_\cdot} \mathfrak{S} \xrightarrow{\pi} M$ the deformation family of the semi-simplicial hyper-resolution $\pi_\cdot : X_\cdot \longrightarrow S$ arising from a locally trivial family of algebraic surfaces with ordinary singularities $(\mathfrak{S},\pi,M)$.

2.2 Theorem([5]). *Let $(\mathfrak{S},\pi,M)$ be a locally trivial family of algebraic surfaces with ordinary singularities, parametrized by a complex manifold M. We define $R^\ell_\mathbb{Z}(\pi) := R^\ell\pi_*\mathbb{Z}_\mathfrak{S}$ (modulo torsion) $(0 \le \ell \le 4)$, $R^\ell_\mathbb{Q}(\pi) := R^\ell_\mathbb{Z}(\pi) \otimes_\mathbb{Z} \mathbb{Q}$ and $R^\ell_\mathcal{O}(\pi) := R^\ell\pi_*(\pi^\cdot\mathcal{O}_M) \simeq \mathbb{R}^\ell\pi_*(DR^\cdot_{\mathfrak{S}/M})$, where $\pi^\cdot\mathcal{O}_M$ is the topological inverse of the structure sheaf of M by the map $\pi : \mathfrak{S} \longrightarrow M$ and $DR^\cdot_{\mathfrak{S}/M}$ the cohomological relative de Rham complex of the family $\pi : \mathfrak{S} \longrightarrow M$. Then there* *exist a family of increasing sub-local systems* $\mathbb{W}$ *(weight filtration) on* $R^{\ell}_{\mathbb{Q}}(\pi)$ *and a family of decreasing holomorphic subbundles* $\mathbb{F}$ *(Hodge filtration) on* $R^{\ell}_{\mathcal{O}}(\pi)$ *such that* $\{R^{\ell}_{\mathbb{Z}}(\pi), (R^{\ell}_{\mathbb{Q}}(\pi), \mathbb{W}[\ell]), (R^{\ell}_{\mathcal{O}}(\pi), \mathbb{W}[\ell], \mathbb{F})\}$ *is a variation of mixed Hodge structures , where* $\mathbb{W}[\ell]$ *denotes the shift of the filtration degree to the right by* ℓ, *i.e.,* $\mathbb{W}[\ell]_q := \mathbb{W}_{q-\ell}$. *Furthermore, there exist spectral sequences*

$$
{}_W E_1^{p,q} \simeq R^q(\pi \circ \varpi_p)_* \mathbb{Q}_{\mathfrak{X}_p} \Rightarrow {}_W E_\infty^{p,q} = Gr^W_{-p}(R^{p+q}_{\mathbb{Q}}(\pi)),
$$
$$
{}_F E_1^{p,q} \simeq \mathbb{R}^q(\pi \circ \varpi.)_* \Omega^p_{\mathfrak{X}./M} \Rightarrow {}_F E_\infty^{p,q} = Gr^p_F(R^{p+q}_{\mathcal{O}}(\pi))
$$

with ${}_W E_2^{p,q} = {}_W E_\infty^{p,q}$, ${}_F E_1^{p,q} = {}_F E_\infty^{p,q}$.

§3 Infinitesimal mixed Torelli problem

Let S be a complex projective surface with ordinary sungularities and let $(\mathfrak{S}, \pi, (M, o))$ be a locally trivial deformation family of S parametrized by a pointed complex manifold (M, o). Since $(R^{\ell}_{\mathbb{Z}}(\pi), R^{\ell}_{\mathbb{Q}}(\pi), R^{\ell}_{\mathcal{O}}(\pi))$ forms a variation of pure (resp. mixed) Hodge structures for $\ell = 1$ (resp. $\ell = 2$), we have a holomorphic map (so-called *period map*)

$$
\Phi := \Phi_1 \times \Phi_2 : M \longrightarrow \mathcal{M}(H^1(S)_{\mathbb{Z}}) \times \mathcal{M}_{mix}(H^2(S)_{\mathbb{Z}})
$$

at least from a sufficiently small open neighborhood of o in M to the product of the *modular variety* of pure Hodge structures on $H^1(S)_{\mathbb{Z}} := H^1(S, \mathbb{Z})$ (modulo torsion) and that of mixed Hodge structures on $H^2(S)_{\mathbb{Z}} := H^2(S, \mathbb{Z})$ (modulo torsion). In the sequel we always work in a a sufficiently small open neighborhood of o in M.

<Infinitesimal mixed Torelli problem for S >

If the characteristic map (=Kodaira-Spencer map) $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$ ($\Theta_S := \mathcal{H}om_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S)$ =the sheaf of germs of holomorphic vector fields on S, T_oM= the tangent space of M at o) *of the family* $(\mathfrak{S}, \pi, (M, o))$ *is injective , then is the Jacobian map*

$$
d\Phi_o : T_oM \longrightarrow T_{\Phi_1(o)}(\mathcal{M}(H^1(S)_{\mathbb{Z}})) \oplus T_{\Phi_2(o)}(\mathcal{M}_{mix}(H^2(S)_{\mathbb{Z}}))
$$

injective ?

The relation between the Jacobian map $(d\Phi_\ell)_o$, $\ell = 1, 2$, and the characteristic map $\sigma_o : T_oM \longrightarrow H^1(S, \Theta_S)$ of the family $(\mathfrak{S}, \pi, (M, o))$ at the point o is given by the following theorem.

3.1 Theorem ([5]).

(i) For $\ell = 1, 2$, *the following diagram commutes up to* $\oplus_{p=1}^{\ell}(-1)^{p+1}$:

$$
\begin{array}{ccc}
T_oM & \longrightarrow & \oplus_{p=1}^{\ell} Hom_{\mathbb{C}}(\mathbb{H}^{\ell-p}(\Omega^p_{X.}), \mathbb{H}^{\ell-p+1}(\Omega^{p-1}_{X.})) \\
{\scriptstyle \rho_o} \searrow & & \nearrow {\scriptstyle \text{coupling by the "contraction"}} \\
 & H^1(S, \Theta(\pi.)) &
\end{array}
$$

(ii) If $H^0(D_X^*, \Theta_{D_X^*}(-\Sigma t_X^*)) = 0$*, then*

$$H^1(S, \Theta_S) \simeq H^1(S, \Theta(\pi.)),$$

where $\mathbb{H}^*(\Omega^p_{X.})$ *denote the hypercohomology of the semi-simplicial object* $\Omega^p_{X.}$, $\Theta(\pi.)$ *the sheaf of germs of holomorphic tangent vector fields of the semi-simplicial hyper-resolution* $\pi. : X. \to S$*, and* $\rho_o : T_oM \to H^1(S, \Theta(\pi.))$ *is the characteristic map of the deformation family* $\mathfrak{X}. \xrightarrow{\varpi.} \mathfrak{S} \xrightarrow{\pi} M$.

<Cohomological infinitesimal mixed Torelli problem for S >

Is the homomorphism

$$H^1(S, \Theta(\pi.)) \to \oplus_{\ell=1}^2 \{\oplus_{p=1}^{\ell} Hom_{\mathbb{C}}(\mathbb{H}^{\ell-p}(\Omega^p_{X.}), \mathbb{H}^{\ell-p+1}(\Omega^{p-1}_{X.}))\}$$

defined by the "contraction" injective ?

3.2 Theorem ([7]). *The cohomological infinitesimal mixed Torelli holds for* S *if all of the following conditions are satisfied:*

(i) The infinitesimal Torelli holds for X*, that is, the homomorphism*

$$H^1(X, \Theta_X) \to \oplus_{p=1}^2 Hom_{\mathbb{C}}(H^0(X, \Omega^p_X), H^1(X, \Omega^{p-1}_X))$$

by the "contraction" is injective.

(ii) The homomorphism

$$H^0(D_X, \mathcal{N}_{D_X/X})/Im\ \{H^0(X, \Theta_X) \to H^0(D_X, \mathcal{N}_{D_X/X})\}$$
$$\to Hom_{\mathbb{C}}(H^0(X, \Omega^2_X), H^0(D_X^*, \Omega^1_{D_X^*})/Im\ \{H^0(D_S^*, \Omega^1_{D_S^*}) \oplus H^0(X, \Omega^1_X)\})$$

defined by the "contraction" and the "pull-back" is injective, where $\mathcal{N}_{D_X/X} := \Theta_X/\Theta_X(-logD_X)$.

(iii) $H^0(D_X^*, g^*(\Theta_{D_S^*}(-\Sigma t_S^*))) = 0$.

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