Global existence of weak solutions to general quasilinear degenerate Keller-Segel systems*

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1. Introduction

In this talk we consider the global existence of weak solutions to the following Cauchy problem for general quasilinear degenerate Keller-Segel systems:

$$\text{(KS)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (f(u)\nabla u - g(u)\nabla v), & \text{in } \mathbb{R}^N \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $u, v : \mathbb{R}^N \times [0,\infty) \to [0,\infty)$ are unknown functions, and we assume that $N \in \mathbb{N}$, $u_0, v_0 \geq 0$, $f \in C([0,\infty)) \cap C^1((0,\infty))$, $g \in C^1([0,\infty))$, $f, g > 0$ on $(0,\infty)$ and

$$f(0) = g(0) = 0.$$

In particular, if $f(u) = 1$ and $g(u) = u$, then (KS) is the so-called (minimal) Keller-Segel model proposed by [**4**] in 1970. The model describes a part of cellular slime molds with the chemotaxis at the life cycle. Usually $u(x,t)$ shows the density of cellular slime molds and $v(x,t)$ shows the density of the semiochemical at place x and time t. There are biological and mathematical generalizations of the Keller-Segel model. For example, Sugiyama [**5**] and Sugiyama-Kunii [**6**] started mathematical studies on the global existence of weak solutions to (KS) with $f(u) = u^{m-1}$ and $g(u) = u^{q-1}$; note that such porous medium-type diffusion is motivated from a biological point of view (see [**7**]) and nonlinear diffusion has been suggested by Hillen-Painter [**1**].

In general, the study on quasilinear degenerate parabolic systems is a challenging issue. To study (KS), we encounter the following difficulties. Firstly, the well known theory for non-degenerate parabolic equations (existence theorem, maximum principle, etc.) can not be applied directly to (KS). Even if $f(u)$ is replaced with the approximation $f(u+\varepsilon)$, the results depend strongly on $\varepsilon > 0$. Secondly, there is no representation formula for the solution u, and hence it is very difficult to obtain apriori estimates of solutions.

The purpose of this talk is to establish the global existence of weak solutions to (KS) (for the definition of weak solutions see Section 2). More precisely, we generalize the following two existence results for (KS) with $f(u) = u^{m-1}$ $(m \geq 1)$ and $g(u) = u^{q-1}$ $(q \geq 2)$ which were recently obtained by [**2**] and [**3**], respectively:

- if $q < m + 2/N$, then (KS) has a global weak solution without any restriction on the size of initial data;
- if $q \geq m + 2/N$, then (KS) has a global weak solution for small initial data.

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2. Main results

Before stating our result we define weak solutions to (KS).

Definition 1. Let $T > 0$. A pair (u, v) of non-negative functions defined on $\mathbb{R}^N \times (0, T)$ is called a *weak solution* to (KS) *on* $[0, T)$ if

(a) $u \in L^\infty(0, T; L^p(\mathbb{R}^N))$ $(\forall\, p \in [1, \infty])$, $F(u) := \int_0^u f(r)\, dr \in L^2(0, T; H^1(\mathbb{R}^N))$;

(b) $v \in L^\infty(0, T; H^1(\mathbb{R}^N))$;

(c) (u, v) satisfies (KS) in the distributional sense, i.e., for every $\varphi \in C_0^\infty(\mathbb{R}^N \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}^N} (\nabla F(u) \cdot \nabla\varphi - g(u)\nabla v \cdot \nabla\varphi - u\varphi_t)\, dxdt = \int_{\mathbb{R}^N} u_0(x)\varphi(x, 0)\, dx,$$
$$\int_0^T \int_{\mathbb{R}^N} (\nabla v \cdot \nabla\varphi + v\varphi - u\varphi - v\varphi_t)\, dxdt = \int_{\mathbb{R}^N} v_0(x)\varphi(x, 0)\, dx.$$

We now state the main results.

Theorem 2.1 (the sub-critical case). *Let $N \in \mathbb{N}$ and $T > 0$. Assume that there exist constants $c_0, c_1, c_2 > 0$ and $M \geq m \geq 2$ such that*

$$c_0\, r^{m-1} \leq f(r) \leq c_1 r^{M-1} \quad (r \geq 0); \tag{2.1}$$
$$g(r) \leq c_2\, f(r) r^\alpha \quad (r \geq 0) \quad \text{for some } \alpha < 2/N. \tag{2.2}$$

Then for any $u_0, v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\Delta v_0 \in L^{p_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (for some $p_0 > 1$) there exists a non-negative (global) weak solution (u, v) to (KS) *on* $[0, T)$.

Theorem 2.2 (the super-critical case). *Let $N \geq 2$ and $T > 0$. Assume that there exist constants $c_0, c_1, c_3, c_4 > 0$ and $M \geq m \geq 2$ satisfying* (2.1) *and*

$$c_3\, f(r) r^\alpha \leq g(r) \leq c_4\, f(r) r^\alpha \quad (r \geq 0) \quad \text{for some } \alpha \geq 2/N. \tag{2.3}$$

Assume further that $u_0, v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\Delta v_0 \in L^{p_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (for some $p_0 > 1$) and $\|u_0\|_{L^{\frac{N}{2}}}$, $\|u_0\|_{L^{\frac{N\alpha}{2}}}$, $\|\Delta v_0\|_{L^{\frac{N}{2}+1}}$, $\|\Delta v_0\|_{L^{\frac{N\alpha}{2}+1}}$, $\|\Delta v_0\|_{L^{\frac{N}{2}+m+\alpha+1}}$, $\|\Delta v_0\|_{L^{\frac{N\alpha}{2}+m+\alpha+1}}$ are sufficiently small. Then there exists a non-negative (global) weak solution (u, v) to (KS) *on* $[0, T)$.

References

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