A Blow-up Boundary for a Semilinear Wave Equation with Power Nonlinearity

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Caffarelli and Friedman investigated blow-up boundaries for a Cauchy problem of semilinear wave equations with power nonlinearity and showed remarkable results(see [1]). We consider the same problem for initial data with different conditions from theirs.

We consider the following Cauchy problem for a wave equation:

$$\begin{cases} \Box u = (\partial_t^2 - \Delta)u = F(u), \text{ in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = f(x), \ \partial_t u(x,0) = g(x) \text{ in } \mathbb{R}^3. \end{cases} \tag{0.1}$$

where $F(u) = |u|^{p-1}u$ or $F(u) = |u|^p$ with $p > 1$.

The integral equation derived from (0.1):

$$u(x,t) = u_0(x,t) + \frac{1}{4\pi}\int_0^t (t-s)ds \int_{|\omega|=1} \{F(u(x+(t-s)\omega, s))\}d\omega, \tag{0.2}$$

where

$$\begin{aligned} u_0(x,t) =& \frac{\partial}{\partial t}\frac{t}{4\pi}\int_{|\omega|=1} f(x+t\omega)d\omega + \frac{t}{4\pi}\int_{|\omega|=1} g(x+t\omega)d\omega \\ =& \frac{t}{4\pi}\int_{|\omega|=1}\{g(x+t\omega) + \omega\cdot\nabla f(x+t\omega) + \frac{f(x+t\omega)}{t}\}d\omega. \end{aligned} \tag{0.3}$$

The sequence $\{u_n\}, n = 0, 1, \cdots$ is given by

$$u_n(x,t) = u_0(x,t) + \frac{1}{4\pi}\int_0^t (t-s)ds \int_{|\omega|=1} \{F(u_{n-1}(x+(t-s)\omega, s))\}d\omega. \tag{0.4}$$

Then $u(x,t) = \lim_{n\to\infty} u_n(x,t)$.

The blow-up boundary Γ is defined by

$$\Gamma = \partial\{(x,t)/u(x,t) < \infty, \ t > 0\}.$$

We can give several suitable conditions to initial data to show in some domain $K_{R,T} \subset \mathbb{R}^3 \times [0,T)$ that

1. (0.1) has a C^2 positive real-valued local solution u,
2. u is monotone increasing in t for any fixed x and moreover satisfies $\partial_t u_n \geq |\nabla u_n|$,
3. there exists a positive $T(x)$ for any x such that u keeps its regularity in $K_{R,T} \cap \{0 < t < T(x)\}$ and $\lim_{t \nearrow T(x)} u(x,t) = \infty$.

Then the blowup boundary Γ exists and is represented by a function $\phi(x) = T(x)$ satisying that

$$|\phi(x) - \phi(y)| \leq |x - y|. \tag{0.5}$$

We assume the following assumptions to obtain (0.5).

Assumption I

Let $f \in C^3(\mathbf{R}^3)$ and $g \in C^2(\mathbf{R}^3)$, and let R_0 be a positive constant.

$$\begin{cases} f(x) \geq 0 \text{ for } |x| \geq R_0 \\ g(x) + \omega \cdot \nabla f(x) + \frac{f(x)}{1+|x|} \geq 0 \text{ for } |x| \geq R_0, \end{cases} \tag{0.6}$$

where $\omega = x/|x|$.

Assumption III

Let $f \in C^3(\mathbf{R}^3)$ and $g \in C^2(\mathbf{R}^3)$. Let R_0 be a positive constant. Let e be any unit vector in $\mathbf{R}^3$ and $\omega = \frac{x}{|x|}$. Assume for $|x| \geq R_0$

$$\begin{cases} g(x) + e \cdot \nabla f(x) \geq 0, \\ \triangle f(x) + F(f(x)) + e \cdot \nabla g(x) + \omega \cdot \nabla(g(x) + e \cdot \nabla f(x)) \\ + \frac{g(x)+e\cdot\nabla f(x)}{1+|x|} \geq 0. \end{cases} \tag{0.7}$$

References

[1] L.A.Caffarelli and A.Friedman, The blow-up boundary for nonlinear wave equations, *Trans.AMS.* **297** (1986), 223–241.

[2] H.Uesaka, The blow-up boundary for a system of semilinear wave equations, in Proceedings of the 6th ISAAC, (2009).