On the Wavelets Having Gevrey Regularities and Subexponential Decays

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An MRA wavelet ψ is determined by a scaling function φ as

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{m\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right).$$

Meyer found orthonormal wavelets having polynomial decays or especially subexponential decays. Actually, the subexponential decay implies the degeneracy of the order of Gevrey type. In this talk, we shall define the subexponential decay as follows:

Definition *Let $s > 1$. A function f is called to have a subexponential decay of order s, written $f \in \Gamma^s(\mathbf{R}_x)$, if there exist some $C > 0$ and $\rho > 0$ such that*

$$|f(x)| \le C \exp\left[-\rho |x|^{\frac{1}{s}}\right].$$

This kind of estimate is used in the frequency domain with the well-known Paley-Wiener theorem and gives the regularity in the time domain. Meyer constructed $\hat{\varphi}(\xi)$, which belongs to the Gevrey class in the frequency domain, for any compact set $K \subset \mathbf{R}_x$

$$\sup_{\xi \in K}\left|\partial_\xi^n \hat{\varphi}(\xi)\right| \le C_K R^n n!^s \quad \text{for } n \in \mathbf{N} \quad (C_K > 0,\ R > 0).$$

The regularity of $\hat{\varphi}(\xi)$ comes from just the regularity of the low-pass filter $m(\xi)$. However, it would be difficult to control the decay rate of $\hat{\varphi}$ except the band-limited case. Indeed, such a wavelet seems not to be found; there are two blanks in the following table.

	A_x	G_x^s	C_x^r
A_ξ	nonexistence	nonexistence	Battle-Lemarié, Daubechies
G_ξ^s	Meyer		HWW
C_ξ^r	Meyer		HWW

In the above table HWW denotes the wavelets introduced by E. Hernández, X. Wang and G. Weiss [3]. Here the word "nonexistence" is shown by the following theorem (see [1], [2], etc.):

Theorem A *There is no orthonormal wavelet belonging to C_x^∞ and having an exponential decay.*

The Daubechies type avoids this restriction by relaxing the regularity C_x^∞ and thus attains $\mathcal{A}_\xi$, especially, compact-support in the time domain. Our strategy is to relax the regularity $\mathcal{A}_\xi$ in the frequency domain and seek orthonormal wavelets having regularities beyond C_x^∞. We shall construct new orthonormal wavelets which fill in the blanks of the table, i.e., new wavelets having Gevrey regularities both in time and frequency.

Main Theorem *Let $s^* > 1$. There exists a wavelet ψ satisfying both $\psi \in G_x^s$ and $\hat{\psi} \in G_\xi^{s^*}$ for*

$$s = \max\{1, s^* - 1\}.$$

References

[1] *Ingrid Daubechies. Ten Lectures on Wavelets.* Society for Industrial Mathematics, 1992.

[2] *Jacek Dzibański; Eugenio Hernàndez. Band-Limited Wavelets with Subexponential Decay.* Canad. Math. Bull. Vol. 41, No. 4, 398-403 (1998).

[3] *Eugenio Hernàndez; Xihua Wang; Guido Weiss. Smoothing Minimally Supported Frequency Wavelets: Part II.* The Journal of Fourier Analysis and Applications, Vol. 1, No. 1, 23-41 (1997).

[4] *Shinya Moritoh; Kyoko Tomoeda. A Further Decay Estimate for the Dzibański-Hernàndez Wavelets.* Canad. Math. Bull. Vol. 53, No. 1, 133-139 (2010).