On second order weakly hyperbolic equations and the ultradifferentiable classes

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1 Introduction

We study the Cauchy problem for second order weakly hyperbolic equations with time dependent coefficients:

$$\begin{cases} \left(\partial_t^2 - a(t)^2\Delta\right) u = 0, \quad (t,x) \in (0,T] \times \mathbf{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbf{R}^n, \end{cases} \tag{1}$$

where $a(t) \geq 0$ and $T > 0$. The pioneer paper [1] shows that the *smoothness* of the coefficient $a(t)$ is crucial for the well-posedness of (1). Precisely, if $a(t)$ is smoother in the sense of Hölder continuity, then (1) is well-posed in the Gevrey class of larger order. Furthermore, it is studied in [2] that (1) is well-posed in the appropriate functions space, which is set between C^∞ class and the Gevrey class, if $a(t)$ belongs to an intermediate class between C^∞ and real analytic class. In particular, it is studied in [4] that if $a(t) > 0$ on $[0,T)$ and $a(T) = 0$ then (1) can be C^∞ well-posed for $a(t) \in C^2$ under suitable assumptions to $a(t)$ for the orders of *degeneration* and *oscillation* as $t \to T$. Generally, we cannot expect that further smoothness of $a(t)$ than C^2 brings a benefit as the results [1, 2] for the model of one point degeneration. However, we can do it if we introduce an additional property of the coefficient, which is called the *stabilization* property. Indeed, it is proved in [3] that there exists an example of $a(t)$ such that the C^∞ well-posedness cannot be proved by [4] but can be done if we assume a suitable stabilization condition and $a(t) \in C^\infty$ simultaneously. The main purpose of this talk is to consider a possibility that further smoothness of $a(t)$, which belongs to the ultradifferentiable class, bring a benefit for the C^∞ well-posedness of (1).

2 Main theorem

For a function $\lambda(t)$ satisfying

$$\lambda'(t) \leq 0, \;\; \lambda(t) > 0 \;\text{ on }\; [0,T) \;\text{ and }\; \lambda(T) = 0 \tag{2}$$

we denote

$$\Lambda(t) = \int_t^T \lambda(s)\,ds. \tag{3}$$

Moreover, we define the positive monotone decreasing function $\Theta(t)$ by

$$\Theta(t) = \int_t^T |a(s) - \lambda(s)|\,ds. \tag{4}$$

Let us introduce the following hypothesis:

(H1) There exists a constant $C_0 > 1$ such that

$$C_0^{-1}\lambda(t) \leq a(t) \leq C_0\lambda(t). \tag{5}$$

(H2)

$$\Theta(t) = o(\Lambda(t)) \quad (t \to T). \tag{6}$$

(H3) $a(t) \in C^\infty([0,\infty))$ satisfies

$$\frac{|a^{(k)}(t)|}{\lambda(t)} \le M_k \rho(t)^k \quad (k=0,1,\ldots) \tag{7}$$

for a positive and strictly increasing function $\rho(t) \in C^0([0,T))$ satisfying $\lim_{t\to T}\rho(t)=\infty$, and a sequence of positive real numbers $\{M_k\}_{k=0}^\infty$ satisfying

$$\frac{M_k}{kM_{k-1}} \le \frac{M_{k+1}}{(k+1)M_k} \quad (k=1,2,\ldots). \tag{8}$$

Here we define the associated function $\mathcal{M}(r)$ of $\{M_k\}$ by

$$\mathcal{M}(r)=\mathcal{M}(r;\{M_k\}) := \sup_{k\ge 1}\left\{\frac{r^k}{M_k}\right\} \quad (r>0). \tag{9}$$

Then our main theorem is represented as follows:

Theorem 1. *If there exist $\lambda(t)\in C^1([0,T])$ satisfying (2) and a sequence $\{M_k\}_{k=0}^\infty$ such that (H1), (H2) and (H3) with*

$$\rho(t)=\frac{\lambda(t)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t)}{\Theta(t)}\right)}\sigma\left(\frac{1}{\Theta(t)}\right) \tag{10}$$

are valid for a positive strictly increasing function $\sigma(r)\in C^0([0,\infty))$, then there exists a positive constant C such that the following estimate is established

$$\|(u(t,\cdot),u_t(t,\cdot))\|_{L^2} \le \left\|\mathrm{e}^{C\mu(\langle D\rangle)}\left(u_0(\cdot),u_1(\cdot)\right)\right\|_{L^2}, \tag{11}$$

where

$$\mu(\tau)=\frac{\tau}{\sigma^{-1}(\tau)}. \tag{12}$$

References

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