

5章 三角関数

問 1

$$\begin{aligned} \sin 75^\circ &= \sin(45^\circ + 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} \tan 75^\circ &= \tan(45^\circ + 30^\circ) \\ &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1 \cdot \frac{1}{\sqrt{3}}} \\ &= \frac{\left(1 + \frac{1}{\sqrt{3}}\right) \times \sqrt{3}}{\left(1 - \frac{1}{\sqrt{3}}\right) \times \sqrt{3}} \\ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \\ &= \frac{(\sqrt{3} + 1)^2}{(\sqrt{3} - 1)(\sqrt{3} + 1)} \\ &= \frac{3 + 2\sqrt{3} + 1}{3 - 1} \\ &= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3} \end{aligned}$$

〔別解〕

$$\begin{aligned} \tan 75^\circ &= \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\frac{\sqrt{6} + \sqrt{2}}{4}}{\frac{\sqrt{6} - \sqrt{2}}{4}} \\ &= \frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} = \frac{(\sqrt{6} + \sqrt{2})^2}{(\sqrt{6} - \sqrt{2})(\sqrt{6} + \sqrt{2})} \\ &= \frac{6 + 2 \cdot 2\sqrt{3} + 2}{6 - 2} \\ &= \frac{8 + 4\sqrt{3}}{4} = 2 + \sqrt{3} \end{aligned}$$

$$\begin{aligned} \sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} \cos 15^\circ &= \cos(45^\circ - 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} + \sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} \tan 15^\circ &= \tan(45^\circ - 30^\circ) \\ &= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} \\ &= \frac{\left(1 - \frac{1}{\sqrt{3}}\right) \times \sqrt{3}}{\left(1 + \frac{1}{\sqrt{3}}\right) \times \sqrt{3}} \\ &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \\ &= \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)} \\ &= \frac{3 - 2\sqrt{3} + 1}{3 - 1} \\ &= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3} \end{aligned}$$

〔別解〕

$$\begin{aligned} \tan 15^\circ &= \frac{\sin 15^\circ}{\cos 15^\circ} = \frac{\frac{\sqrt{6} - \sqrt{2}}{4}}{\frac{\sqrt{6} + \sqrt{2}}{4}} \\ &= \frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} = \frac{(\sqrt{6} - \sqrt{2})^2}{(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})} \\ &= \frac{6 - 2 \cdot 2\sqrt{3} + 2}{6 - 2} \\ &= \frac{8 - 4\sqrt{3}}{4} = 2 - \sqrt{3} \end{aligned}$$

問 2

$$\begin{aligned} \text{与式} &= \frac{\tan \theta + \tan \frac{\pi}{4}}{1 - \tan \theta \tan \frac{\pi}{4}} \\ &= \frac{\tan \theta + 1}{1 - \tan \theta \cdot 1} \\ &= \frac{1 + \tan \theta}{1 - \tan \theta} \end{aligned}$$

問 3

α は第 2 象限の角だから, $\cos \alpha < 0$

よって

$$\begin{aligned}\cos \alpha &= -\sqrt{1 - \sin^2 \alpha} \\ &= -\sqrt{1 - \left(\frac{1}{\sqrt{3}}\right)^2} \\ &= -\sqrt{1 - \frac{1}{3}} \\ &= -\sqrt{\frac{2}{3}} = -\frac{\sqrt{6}}{3}\end{aligned}$$

β は第 4 象限の角だから, $\cos \beta > 0$

よって

$$\begin{aligned}\cos \beta &= \sqrt{1 - \sin^2 \beta} \\ &= \sqrt{1 - \left(-\frac{\sqrt{5}}{6}\right)^2} \\ &= \sqrt{1 - \frac{5}{36}} \\ &= \sqrt{\frac{31}{36}} = \frac{\sqrt{31}}{6}\end{aligned}$$

したがって

$$\begin{aligned}\sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{31}}{6} - \left(-\frac{\sqrt{6}}{3}\right) \cdot \left(-\frac{\sqrt{5}}{6}\right) \\ &= \frac{\sqrt{93}}{18} - \frac{\sqrt{30}}{18} \\ &= \frac{\sqrt{93} - \sqrt{30}}{18}\end{aligned}$$

$$\begin{aligned}\cos(\alpha - \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= -\frac{\sqrt{6}}{3} \cdot \frac{\sqrt{31}}{6} + \frac{\sqrt{3}}{3} \cdot \left(-\frac{\sqrt{5}}{6}\right) \\ &= -\frac{\sqrt{186}}{18} - \frac{\sqrt{15}}{18} \\ &= -\frac{\sqrt{186} + \sqrt{15}}{18}\end{aligned}$$

問 4

$0 < \alpha < \frac{\pi}{2}$, $0 < \beta < \frac{\pi}{2}$ の辺々を加えると

$$0 < \alpha + \beta < \frac{\pi}{2} + \frac{\pi}{2}$$

すなわち, $0 < \alpha + \beta < \pi \cdots \textcircled{1}$

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ &= \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} \\ &= \frac{\frac{5}{6}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1\end{aligned}$$

また, ①より, $\alpha + \beta = \frac{\pi}{4}$

問 5

α は第 2 象限の角だから, $\sin \alpha > 0$

よって

$$\begin{aligned}\sin \alpha &= \sqrt{1 - \cos^2 \alpha} \\ &= \sqrt{1 - \left(-\frac{4}{5}\right)^2} \\ &= \sqrt{1 - \frac{16}{25}} \\ &= \sqrt{\frac{9}{25}} = \frac{3}{5}\end{aligned}$$

したがって

$$\begin{aligned}\sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= 2 \cdot \frac{3}{5} \cdot \left(-\frac{4}{5}\right) \\ &= -\frac{24}{25} \\ \cos 2\alpha &= 2 \cos^2 \alpha - 1 \\ &= 2 \cdot \left(-\frac{4}{5}\right)^2 - 1 \\ &= \frac{32}{25} - 1 = \frac{7}{25}\end{aligned}$$

$$\begin{aligned}\tan 2\alpha &= \frac{\sin 2\alpha}{\cos 2\alpha} \\ &= \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}\end{aligned}$$

問 6

$$\begin{aligned}\cos^2 \frac{\pi}{8} &= \cos^2 \frac{\frac{\pi}{4}}{2} \\ &= \frac{1 + \cos \frac{\pi}{4}}{2} \\ &= \frac{1 + \frac{1}{\sqrt{2}}}{2} \\ &= \frac{\left(1 + \frac{\sqrt{2}}{2}\right) \times 2}{2 \times 2} \\ &= \frac{2 + \sqrt{2}}{4}\end{aligned}$$

$\cos \frac{\pi}{8} > 0$ であるから

$$\cos \frac{\pi}{8} = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

問 7

$\frac{\pi}{2} < \alpha < \pi$ より, $\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2} \dots \textcircled{1}$

$$\begin{aligned} \sin^2 \frac{\alpha}{2} &= \frac{1 - \cos \alpha}{2} \\ &= \frac{1 - \left(-\frac{1}{9}\right)}{2} \\ &= \frac{\frac{10}{9}}{2} = \frac{5}{9} \end{aligned}$$

$\textcircled{1}$ より, $\sin \frac{\alpha}{2} > 0$ であるから

$$\sin \frac{\alpha}{2} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

また

$$\begin{aligned} \cos^2 \frac{\alpha}{2} &= \frac{1 + \cos \alpha}{2} \\ &= \frac{1 + \left(-\frac{1}{9}\right)}{2} \\ &= \frac{\frac{8}{9}}{2} = \frac{4}{9} \end{aligned}$$

$\textcircled{1}$ より, $\cos \frac{\alpha}{2} > 0$ であるから

$$\begin{aligned} \cos \frac{\alpha}{2} &= \sqrt{\frac{4}{9}} = \frac{2}{3} \\ \tan \frac{\alpha}{2} &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \\ &= \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} = \frac{\sqrt{5}}{2} \end{aligned}$$

問 8

$$\begin{aligned} (1) \text{ 与式} &= \frac{1}{2} \{ \cos(3\theta + 5\theta) + \cos(3\theta - 5\theta) \} \\ &= \frac{1}{2} \{ \cos 8\theta + \cos(-2\theta) \} \\ &= \frac{1}{2} (\cos 8\theta + \cos 2\theta) \end{aligned}$$

$$\begin{aligned} (2) \text{ 与式} &= -\frac{1}{2} \{ \cos(7\theta + 2\theta) - \cos(7\theta - 2\theta) \} \\ &= -\frac{1}{2} (\cos 9\theta - \cos 5\theta) \\ &= \frac{1}{2} (\cos 5\theta - \cos 9\theta) \end{aligned}$$

問 9

$$\begin{aligned} (1) \text{ 与式} &= 2 \sin \frac{4\theta + 2\theta}{2} \cos \frac{4\theta - 2\theta}{2} \\ &= 2 \sin \frac{6\theta}{2} \cos \frac{2\theta}{2} \\ &= 2 \sin 3\theta \cos \theta \end{aligned}$$

$$\begin{aligned} (2) \text{ 与式} &= 2 \cos \frac{3\theta + 5\theta}{2} \cos \frac{3\theta - 5\theta}{2} \\ &= 2 \cos \frac{8\theta}{2} \cos \frac{-2\theta}{2} \\ &= 2 \cos 4\theta \cos(-\theta) \\ &= 2 \cos 4\theta \cos \theta \end{aligned}$$

問 10

$$\begin{aligned} (1) \quad y &= \sqrt{1^2 + 1^2} \sin(x + \alpha) \\ &= \sqrt{2} \sin(x + \alpha) \end{aligned}$$

ここで, $\cos \alpha = \frac{1}{\sqrt{2}}$, $\sin \alpha = \frac{1}{\sqrt{2}}$ より, $\alpha = \frac{\pi}{4}$

よって, $y = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$

$$\begin{aligned} (2) \quad y &= \sqrt{1^2 + (\sqrt{3})^2} \sin(x + \alpha) \\ &= 2 \sin(x + \alpha) \end{aligned}$$

ここで, $\cos \alpha = \frac{1}{2}$, $\sin \alpha = -\frac{\sqrt{3}}{2}$ より, $\alpha = -\frac{\pi}{3}$

よって, $y = 2 \sin\left(x - \frac{\pi}{3}\right)$

問 11

$$\begin{aligned} y &= \sqrt{2^2 + 3^2} \sin(x + \alpha) \\ &= \sqrt{13} \sin(x + \alpha) \end{aligned}$$

ただし, $\cos \alpha = \frac{2}{\sqrt{13}}$, $\sin \alpha = \frac{3}{\sqrt{13}}$ である.

ここで, $-1 \leq \sin(x + \alpha) \leq 1$ であるから

$$-\sqrt{13} \leq \sqrt{13} \sin(x + \alpha) \leq \sqrt{13}$$

よって, 最大値は $\sqrt{13}$, 最小値は $-\sqrt{13}$