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Recovery in quantum error correction for general noise without measurement

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It is known that one can do quantum error correction without syndrome measurement, which is often done in operator quantum error correction (OQEC). However, the physical realization could be challenging, especially when the recovery process involves high-rank projection operators and a superoperator. We use operator theory to improve OQEC so that the implementation can always be done by unitary gates followed by a partial trace operation. Examples are given to show that our error correction scheme outperforms the existing ones in various scenarios.

Keywords: quantum error correction, operator quantum error correction, higher rank numerical range, mixed unitary channel
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1 Introduction

Quantum systems are vulnerable to disturbance from an external environment, which can lead to decoherence in the system. We have to overcome this difficulty in order to realize a working quantum computer and dependable quantum information processing. Quantum error correction (QEC) [1, 2, 3] is one of the most promising candidates for overcoming decoherence.

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QEC proposals to date are separated roughly into two classes: one employs extra ancilla qubits for error syndrome readout, while the other, called operator quantum error correction (OQEC), employs high-rank projection operators based on the Knill-Laflamme result; for example, see [2, Theorem 10.1] and its proof, and also [4, 5]. There has been strong interest in constructing practical QEC schemes in actual quantum computing or quantum information processing. The major obstacles for the implementation include the following: the syndrome must be read out by introducing extra ancilla qubits during computing/information processing in the former case, while realization of high-rank projection operators is physically challenging in the latter case.

It was shown in [6] that, for some quantum channels, there exist different QEC schemes in which no syndrome measurements, no syndrome readout ancillas and no projection operators were required. In this scheme, the recovery and decoding operations are combined into a single unitary operation, and the output state is a direct product of a decoded data qubit state and an encoding ancilla state. The data qubit state is reproduced without recovering the codeword, and moreover, one can see from our result and proofs that the projection operation in the Knill-Laflamme condition [2, Theorem 10.1] is automatically built into our output state. The purpose of this paper is to extend the results in [6] to general quantum channels. We show that for any quantum channel there is a unitary recovery operation for which the output state is a tensor product of the data qubit state and an encoding ancilla state. As a result, a decoding scheme can be realized by a unitary operation followed by a partial trace operation. It is worth noting that by a result of Stinespring [7], if the quantum states are represented by density operators acting on a Hilbert space $\mathcal{H}$, then every quantum operation or channel (trace preserving completely positive linear map) can be realized as a dilation of the density operators to density operators acting on a Hilbert space $\mathcal{K}$ followed by a partial trace operation, where $\mathcal{K}$ is usually of much higher dimension. In our scheme, there is no need to do the dilation, and only a unitary similarity transform is required. In some examples, one may use a permutation similarity transform, or a simple circuit diagram to implement the unitary similarity transform. It is also worth noting that there are other automated QEC schemes. For instant, in the scheme described in [8], one needs ancillas for error detection, and thus, the number of the extra qubits is the same as the conventional QECC. Nevertheless, it still requires additional ancilla qubits whereas our scheme does not.

The rest of the paper is organized as follows. We introduce the basic notions of QEC and then prove the main theorem in Section 2. We also give simple examples demonstrating our result and a simplified proof of a theorem given in [2] illustrating that our recovery channel can be used to do correction for many other channels related to ours. Section 3 is devoted to summary and discussions.

2 QEC without Measurement

Denote by $M_{m,n}$ the set of $m \times n$ complex matrices and let $M_n := M_{n,n}$ for simplification. Let $\Phi : M_n \to M_n$ be a generalized quantum channel (i.e., a completely positive linear map without the trace-preserving requirement). Then a k-dimensional subspace $V \subseteq \mathbb{C}^n$ is a quantum error-correcting code for Φ if there is a positive scalar γ and a quantum operation $\Psi : M_n \to M_n$ known as the recovery channel, such that $\Psi \circ \Phi(\rho) = \gamma\rho$ whenever the state (density matrix) ρ satisfies $P\rho P = \rho$, where P is the projection operator onto V. A necessary and sufficient condition for the existence of a quantum error-correcting code was found by Knill and Laflamme [4] (see [2, Theorem 10.1], for example).

Theorem 1 *Let $\Phi : M_n \to M_n$ be a quantum channel with the following operator sum representation*

$$\Phi(\rho) = \sum_{j=1}^{r} F_j \rho F_j^\dagger \,. \tag{1}$$

Suppose $P \in M_n$ is a rank-k orthogonal projection with range space V. The following conditions are equivalent.

(a) *V is a quantum error correcting code for Φ.*

(b) *For $1 \le i, j \le r$, $PF_i^\dagger F_j P = \lambda_{ij} P$ with some complex numbers λ_{ij} so that $[\lambda_{ij}]$ is Hermitian.*

In the context of quantum error correction, the matrices $F_1, \ldots, F_r$ in (1) are known as the error operators associated with the channel Φ; for example see [2, Chapter 10]. The proof of Theorem 10.1 in [2] provides a procedure for constructing a recovery channel Ψ for Φ. The focus of OQEC schemes will be on constructing and implementing the recovery channel Ψ for the given channel Φ without measurement. However, the recovery channel Ψ may be hard to implement as the construction involves projection operators and a superoperator. In this connection, we show in the following that one can compose the quantum channel Φ with a unitary similarity transform so that the output state is a direct sum of the zero operator and a tensor product of the decoded data qubit state and an encoding ancilla state. In particular, a simple construction for recovery operators is proposed when n is a multiple of k, which is often the case in the context of quantum error correction with n and k being powers of 2.

Theorem 2 *Let $\Phi : M_n \to M_n$ be a quantum channel of the form in (1). Suppose the equivalent conditions in Theorem 1 hold and $P = WW^\dagger$ with $W^\dagger W = I_k$ so that a density matrix $\rho \in M_n$ satisfying $P\rho P = \rho$ has the form $W\tilde{\rho}W^\dagger$ with $\tilde{\rho} \in M_k$. Then there is an $R \in U(n)$ and a positive definite matrix $\xi \in M_q$ with $q \le \min\{r, n/k\}$ such that for any density matrix $\tilde{\rho} \in M_k$ and $\rho = W\tilde{\rho}W^\dagger \in M_n$, we have*

$$R^\dagger \,\Phi(\rho)\, R = (\xi \otimes \tilde{\rho}) \oplus 0_{n-qk}.$$

In particular, if k divides n so that M_n can be regarded as $M_{n/k} \otimes M_k$, then

$$R^\dagger \,\Phi(\rho)\, R = \tilde{\xi} \otimes \tilde{\rho} \quad \textit{with} \quad \tilde{\xi} = \xi \oplus 0_{n/k-q}$$

and a recovery channel can be constructed as the map $\Psi : M_n \to M_n$ defined by

$$\Psi(\rho') = W(\,\mathrm{tr}_1(R^\dagger \rho' R)\,)W^\dagger,$$

where tr_1 stands for the partial trace over the encoding ancilla Hilbert space. If Φ is trace preserving, i.e., $\sum_{j=1}^{r} F_j^\dagger F_j = I_n$, then $\mathrm{tr}\,\xi = 1$ so that Ψ is also trace preserving.

Proof. Suppose the equivalent conditions in Theorem 1 hold, i.e., $PF_i^\dagger F_j P = \lambda_{ij} P$ for some $\lambda_{ij} \in \mathbb{C}$. Notice that $\Lambda = [\lambda_{ij}]$ is an $r \times r$ positive semi-definite matrix. Suppose Λ has rank

q. Then there is a $U = [u_{ij}] \in U(r)$ and a positive semi-definite matrix $\hat{\xi} = \left[\hat{\xi}_{ij}\right] \in M_r$ such that $U^\dagger \Lambda U = \hat{\xi}$ and $\hat{\xi}_{ij} = 0$ for all $q < i \le r$ or $q \le j \le r$. Equivalently, $\hat{\xi} = \xi \oplus 0_{r-q}$ for some positive definite matrix $\xi = [\xi_{ij}] \in M_q$. Define

$$\tilde{F}_j = \sum_{i=1}^{r} u_{ij} F_i \quad \text{ for } j = 1, \ldots, r.$$

Let

$$F = [\, F_1 \quad F_2 \quad \cdots \quad F_r \,]$$

be an $n \times rn$ matrix obtained by a juxtaposition of $\{F_j\}_{1 \le j \le r}$ in the given order. Similarly, write $\tilde{F} = [\, \tilde{F}_1 \quad \tilde{F}_2 \quad \cdots \quad \tilde{F}_r \,]$. Then $\tilde{F} = F(U \otimes I_n)$ and for any $\rho \in M_n$,

$$\begin{aligned} \Phi(\rho) &= \sum_{j=1}^{r} F_j \rho F_j^\dagger = F(I_r \otimes \rho) F^\dagger = F(U \otimes I_n)(I_r \otimes \rho)(U \otimes I_n)^\dagger F^\dagger \\ &= \tilde{F}(I_r \otimes \rho)\tilde{F}^\dagger = \sum_{j=1}^{r} \tilde{F}_j \rho \tilde{F}_j^\dagger. \end{aligned}$$

So $\Phi(\rho) = \sum \tilde{F}_j \rho \tilde{F}_j^\dagger$ is another operator sum representation of Φ. Furthermore,

$$P\tilde{F}_i^\dagger \tilde{F}_j P = \sum_{k,l=1}^{r} u_{ki}^* u_{lj} P F_k^\dagger F_l P = \sum_{k,l=1}^{r} u_{ki}^* u_{lj} \lambda_{kl} P = \hat{\xi}_{ij} P \quad \text{for all} \quad i, j = 1, \ldots, r.$$

Without loss of generality, we may assume that $\tilde{F}_j = F_j$ and $PF_i^\dagger F_j P = \xi_{ij} P$ for all $1 \le i, j \le q$. Furthermore, replace the matrix F defined above by $F = [\, F_1 \quad F_2 \quad \cdots \quad F_q \,]$. Since $P = WW^\dagger$ with $W^\dagger W = I_k$, it follows that

$$W^\dagger F_i^\dagger F_j W = \xi_{ij} I_k \quad \text{ which is equivalent to } \quad (I_q \otimes W)^\dagger F^\dagger F (I_q \otimes W) = \xi \otimes I_k.$$

Define an $n \times qk$ matrix

$$R_1 = F(I_q \otimes W)(\xi^{-1/2} \otimes I_k).$$

Then $R_1^\dagger R_1 = I_{qk}$. Take an $n \times (n - qk)$ matrix R_2 such that $R = [\, R_1 \quad R_2 \,] \in U(n)$. Then

$$R^\dagger F(I_q \otimes W) = R^\dagger R_1 (\xi^{1/2} \otimes I_k) = \begin{bmatrix} \xi^{1/2} \otimes I_k \\ 0 \end{bmatrix}.$$

Now for any $\rho \in M_n$ with $P\rho P = \rho$, there exists $\tilde{\rho} \in M_k$ such that $\rho = W\tilde{\rho}W^\dagger$. Since $W^\dagger F_j^\dagger F_j W = \hat{\xi}_{jj} I_k = 0$ and hence $F_j W = 0$ for all $j > q$, $\Phi(\rho)$ can be written as

$$\Phi(\rho) = \sum_{j=1}^{r} F_j (W\tilde{\rho}W^\dagger) F_j^\dagger = \sum_{j=1}^{q} F_j (W\tilde{\rho}W^\dagger) F_j^\dagger = F(I_q \otimes (W\tilde{\rho}W^\dagger)) F^\dagger.$$

It follows that

$$\begin{aligned} R^\dagger \Phi(\rho) R &= R^\dagger F(I_q \otimes W)(I_q \otimes \tilde{\rho})(I_q \otimes W^\dagger) F^\dagger R \\ &= \begin{bmatrix} \xi^{1/2} \otimes I_k \\ 0 \end{bmatrix} (I_q \otimes \tilde{\rho}) \, [\, \xi^{1/2} \otimes I_k \quad 0 \,] = \begin{bmatrix} \xi \otimes \tilde{\rho} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Now if k divides n, we have shown that $\Psi \circ \Phi(\rho) = W\left[\mathrm{tr}_1(R^\dagger \Phi(\rho) R)\right] W^\dagger = W\tilde{\rho}W^\dagger = \rho$ as promised.

Finally, to see that $\sum_{j=1}^q \xi_{jj} = 1$ if $\sum_{j=1}^q F_j^\dagger F_j = I_n$, note that

$$P = P\left(\sum_{j=1}^r F_j^\dagger F_j\right) P = \sum_{j=1}^r P F_j^\dagger F_j P = \left(\sum_{j=1}^r \hat{\xi}_{jj}\right) P = \left(\sum_{j=1}^q \xi_{jj}\right) P.$$

The result follows. □

Note that we have shown that if a channel Φ is correctable, its action on the states ρ satisfying $P\rho P = \rho$ is very simple, namely,

$$\Phi(\rho) = R[(\xi \otimes (W^\dagger \rho W)) \oplus 0]R^\dagger.$$

As a result, we can easily recover ρ from $\Phi(\rho)$. It is worth pointing out several features of our scheme.

First, it is known that a recovery channel is a (trace preserving) completely positive linear map, and such a map can always be realized by a dilation of the basic system to a much larger system, followed by a compression [7]. In contrast, our scheme does not require a dilation of the basic system to a larger system.

Second, suppose one considers the algebra generated by the error operators of the quantum channel describing the decoherence that may affect the quantum computing device, and one obtains a decomposition of the algebra as $(M_s \otimes I_r) \oplus \mathcal{A}$. Then one has a noiseless subsystem of dimension r so that a state of the form $(\xi \otimes \rho) \oplus 0$ will be mapped to a state of the form $(\tilde{\xi} \otimes \rho) \oplus 0$ in which the data state $\rho \in M_r$ is not affected by the quantum channel at all; see [5, 9]. Our result shows that as long as a QECC of dimension r exists, one can construct a unitary operation R such that when one encodes a data state $\rho \in M_r$ by $W\rho W^\dagger$, where $WW^\dagger$ is the orthogonal projection with QECC as its range space, then the quantum channel will send the encoded state to $R(\xi \otimes \rho)R^\dagger$. Thus, one can recover ρ by a unitary operation and discarding of a subsystem. Hence, our encoding and decoding scheme strongly resembles the noiseless subsystem approach, but the use of the algebra generated by the error operators is unnecessary. In fact, if we consider the mixed unitary channel $\rho \mapsto (\rho + U\rho U^\dagger)/2$ with diagonal unitary $U = \mathrm{diag}\,(1, -1, i, -i)$, then the algebra generated by the error operators $I/\sqrt{2}$ and $U/\sqrt{2}$ is the algebra of diagonal matrices. Thus, there is no non-trivial noiseless subsystem. Nonetheless we can find a 2-dimensional QECC (one data qubit), and apply our scheme as shown in the following example.

Example 1 Consider a mixed unitary channel $\Phi(\rho) = (\rho + U\rho U^\dagger)/2$ with diagonal unitary $U = \mathrm{diag}\,(1, -1, i, -i)$. One can find a 2-dimensional QECC, which is spanned by the codewords

$$|\bar{0}\rangle = (|00\rangle + |01\rangle)/\sqrt{2} \quad \text{and} \quad |\bar{1}\rangle = (|10\rangle + |11\rangle)/\sqrt{2}.$$

In this case, the corresponding projection operator is given by $P = WW^\dagger$ with

$$W = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^\dagger.$$

Following the proof of Theorem 2, one can construct the recovery operator R as

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & i \\ 0 & 1 & 0 & -i \end{bmatrix}.$$

Then one can check that for a codeword $\rho = W\tilde{\rho}W^\dagger$ with $\tilde{\rho} \in M_2$, we have

$$R^\dagger \Phi(\rho) R = \frac{1}{2} I_2 \otimes \tilde{\rho}.$$

Third, even though we cannot say that our scheme is always better than other QEC schemes, there are examples of noisy channels in which our scheme is simple to implement; see our recent works [10, 11]. Furthermore, comparing our scheme with the one in the proof of the Knill-Laflamme theorem, one can certainly see the advantage in our result as illustrated in the examples below.

Finally, in Theorem 3 we illustrate that one can use the same encoding and decoding scheme to deal with new quantum channels obtained from the given one whenever the error operators are obtained from linear combinations of the old ones. This allows us to deal with quantum channels with error operators chosen from an infinite set. Theorem 2 was demonstrated for three-, five- and nine-qubit quantum error correcting codes explicitly in [6], see also [12, 13]. It is instructive to work out the simplest example of the three-qubit bit-flip QEC to clarify the theorem in the following.

Example 2 We take a pure state data qubit to simplify the notation. A one-qubit data state $|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$ is encoded with two encoding ancilla qubits as $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, which is an element of the code space V. The projection operator is

$$P = \mathrm{diag}\,(1, 0, 0, 0, 0, 0, 0, 1),$$

which is also written as $P = WW^\dagger$, where

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\dagger.$$

Evidently, $W^\dagger W = I_2$. Let

$$\tilde{\rho} = |\psi_0\rangle\langle\psi_0| = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}. \tag{2}$$

The encoded state is then

$$\rho = W\tilde{\rho}W^\dagger = \begin{bmatrix} |\alpha|^2 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\beta^* \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \vdots & & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha^*\beta & 0 & 0 & 0 & 0 & 0 & 0 & |\beta|^2 \end{bmatrix}. \tag{3}$$

The bit-flip quantum channel is defined as

$$\Phi(\rho) = \sum_{i=0}^{3} F_i \rho F_i^\dagger,$$

where

$$F_0 = \sqrt{p_0}\, I_2 \otimes I_2 \otimes I_2, \quad F_1 = \sqrt{p_1}\, \sigma_x \otimes I_2 \otimes I_2, \quad F_2 = \sqrt{p_2}\, I_2 \otimes \sigma_x \otimes I_2, \quad F_3 = \sqrt{p_3}\, I_2 \otimes I_2 \otimes \sigma_x, \tag{4}$$

with $p_0 + \cdots + p_3 \le 1$. Here σ_i is the i-th Pauli matrix. It is easy to verify that $W^\dagger F_i^\dagger F_j W = p_i \delta_{ij} I_2$. Following the proof of Knill-Laflamme's result, see [2, Theorem 10.1] for example, one can construct the recovery channel $\Psi : M_8 \to M_8$ given by

$$\Psi(\rho') = P\rho' P + (I_8 - P)\rho'(I_8 - P).$$

Then we have $\Psi \circ \Phi(\rho) = \rho$ for all codewords $\rho = W\tilde{\rho}W^\dagger$. However, the quantum channel Ψ is hard to implement as it involves projection operators P and $(I_8 - P)$ and, moreover, Ψ is a superoperator. On the other hand, notice that

$$(I_4 \otimes W)^\dagger F^\dagger F (I_4 \otimes W) = \xi \otimes I_4,$$

where $F = [F_0\ F_1\ F_2\ F_3]$ and $\xi = \mathrm{diag}\,(p_0, p_1, p_2, p_3)$. Following the proof of Theorem 2, let $R_1 = F(I_4 \otimes W)(\xi^{-1/2} \otimes I_2)$. Direct computations yield

$$R_1 = E_{11} + E_{27} + E_{35} + E_{44} + E_{53} + E_{66} + E_{78} + E_{82}, \tag{5}$$

where $\{E_{11}, E_{12}, \ldots, E_{88}\}$ is the standard basis for M_8. Then $R_1^\dagger R_1 = I_8$. The matrix R_2 in the proof of Theorem 2 is vacuous since R_1 is unitary by itself. We denote R_1 as R hereafter. Note that

$$R^\dagger F (I_4 \otimes W) = \xi^{1/2} \otimes I_2.$$

For a codeword $\rho = W\tilde{\rho}W^\dagger$, we have

$$\Phi(\rho) = \sum_{j=0}^{3} F_j (W\tilde{\rho}W^\dagger) F_j^\dagger = F(I_4 \otimes (W\tilde{\rho}W^\dagger))F^\dagger. \tag{6}$$

It follows that

$$R^\dagger \Phi(\rho) R = R^\dagger F (I_4 \otimes W)(I_4 \otimes \tilde{\rho})(I_4 \otimes W^\dagger) F^\dagger R = \xi \otimes \tilde{\rho}. \tag{7}$$

Now the decoded data state $\tilde{\rho}$ appears in the output with no syndrome measurements nor explicit projection. It should be pointed out that the unitary operation R in (5) is independent of the choice of nonnegative numbers p_j. A simple encoding and recovery circuit for 3-qubit bit-flip channel, which encodes and recovers an arbitrary 1 qubit state with two ancilla qubits, was presented in [6]. We also note *en passant* that this QEC was obtained in [12] from different viewpoint based on classical error correction.

Recently, using the same scheme and the techniques of higher rank numerical range, we have shown in [10] that there is a quantum error correction which suppresses fully correlated errors of the form $\sigma_i^{\otimes n}$. It has been proved that n qubit codeword encodes (i) $(n-1)$ data qubit states when n is odd and (ii) $(n-2)$ data qubit states when n is even. Furthermore, it has been proved that one cannot encode $(n-1)$ qubits for even n. This shows that our QEC is optimal in this setting.

In [2, Theorem 10.2], the authors showed that the recovery operation constructed for a given quantum channel Φ in their Theorem 10.1 can be used to correct error of other channels whose error operators are linear combinations of those of Φ. In the following, we show that the recovery channel constructed in Theorem 2 above has the same property. In particular, if R is the unitary matrix constructed for Φ in Theorem 2, then $R^\dagger\tilde{\Phi}(\rho)R$ always have the desired direct sum structure.

Theorem 3 *Suppose R is the unitary matrix given in Theorem 2. If $\tilde{\Phi}$ is another quantum channel $\tilde{\Phi}(\rho) = \sum \tilde{F}_j\rho\tilde{F}_j^\dagger$, where the error operators $\tilde{F}_j$'s are linear combinations of F_j's, then there is a positive definite $\tilde{\xi}$ such that for any density matrix $\tilde{\rho} \in M_k$ and $\rho = W\tilde{\rho}W^\dagger \in M_n$, we have*

$$R^\dagger\,\tilde{\Phi}(\rho)\,R = (\tilde{\xi}\otimes\tilde{\rho})\oplus 0.$$

Proof. We use the same notations as in Theorem 2. Suppose $\tilde{F}_j$'s are linear combinations of F_i's, i.e.,

$$\tilde{F}_j = \sum_{i=1}^{r} t_{ij}F_i \quad \text{for} \quad j = 1,\ldots,s.$$

Recall that $F_jW = 0$ for all $j > q$. Then $\tilde{F}_jW = \sum_{i=1}^{q} t_{ij}F_iW$ for all $j = 1,\ldots,s$. Define a $q\times q$ matrix $T = [t_{ij}]_{1\le i,j\le q}$. For any codeword $\rho = W\tilde{\rho}W^\dagger$, by a similar argument as in the proof of Theorem 2, one can see that

$$\tilde{\Phi}(\rho) = F(TT^\dagger\otimes W\tilde{\rho}W^\dagger)F^\dagger.$$

Then

$$R^\dagger\tilde{\Phi}(\rho)R = (\tilde{\xi}\otimes\tilde{\rho})\oplus 0 \quad \text{where} \quad \tilde{\xi} = \xi^{1/2}TT^\dagger\xi^{1/2}.$$

□

Note that by Theorem 3, for a given quantum channel Φ in operator sum form with error operators $\{F_1,\ldots,F_r\}$, one may choose a set of operators $\{E_1,\ldots,E_m\}$, where $m \le r$, in the simplest from so that the set has the same linear span as $\{F_1,\ldots,F_r\}$. Then construct the new channel $\tilde{\Phi}(\rho) = E_1\rho E_1^\dagger + \ldots + E_m\rho E_m^\dagger$ and the recovery operation. The resulting recovery channel for $\tilde{\Phi}$ corrects the original channel Φ.

Example 3 To illustrate the above remark, consider a quantum channel $\Phi : M_8 \to M_8$ defined as

$$\Phi(\rho) = \sum_{i=0}^{3}\tilde{F}_i\rho\tilde{F}_i^\dagger,$$

where

$$\tilde{F}_0 = \sqrt{\tilde{p}_0}I_2\otimes I_2\otimes I_2, \qquad \tilde{F}_1 = \sqrt{\tilde{p}_1}e^{it_1\sigma_x}\otimes I_2\otimes I_2,$$

$$\tilde{F}_2 = \sqrt{\tilde{p}_2}I_2\otimes e^{it_2\sigma_x}\otimes I_2, \qquad \tilde{F}_3 = \sqrt{\tilde{p}_3}I_2\otimes I_2\otimes e^{it_3\sigma_x},$$

with $t_1, t_2, t_3 \in \mathbb{R}$ and probability $\tilde{p}_j$ such that $\sum_{j=0}^{3}\tilde{p}_j \le 1$. Since $e^{it\sigma_x} = \cos t\, I + i\sin t\,\sigma_x$, we see that $\tilde{F}_0,\ldots,\tilde{F}_3$ are linear combinations of $F_0,\ldots,F_3$ of the three-qubit channel introduced in Example 2. Thus, the recovery channel Ψ constructed previously can be used for this channel also. This channel and the three-qubit channel introduced previously are related as follows. The error operators $\tilde{F}_j$ of the present channel are linear combinations of F_j, defined in (4). More precisely,

$$\tilde{F}_j = \sum_{i=0}^{3} t_{ij} F_i \quad \text{for} \quad j = 0, 1, 2, 3,$$

where

$$T = [t_{ij}] = \begin{bmatrix} \sqrt{\tilde{p}_0/p_0} & \sqrt{\tilde{p}_1/p_0}\,\cos t_1 & \sqrt{\tilde{p}_2/p_0}\,\cos t_2 & \sqrt{\tilde{p}_3/p_0}\,\cos t_3 \\ 0 & i\sqrt{\tilde{p}_1/p_1}\,\sin t_1 & 0 & 0 \\ 0 & 0 & i\sqrt{\tilde{p}_2/p_2}\,\sin t_2 & 0 \\ 0 & 0 & 0 & i\sqrt{\tilde{p}_3/p_3}\,\sin t_3 \end{bmatrix}.$$

Define $\tilde{\rho}$ and ρ as in (2) and (3). Then, similar to the computation in (6) and (7), we have

$$R^\dagger\, \Phi(\rho)\, R = \tilde{\xi} \otimes \tilde{\rho},$$

where $\tilde{\xi}$ is defined by $\xi^{1/2} T T^\dagger \xi^{1/2}$, which is equal to

$$\begin{bmatrix} \tilde{p}_0 + \tilde{p}_1 \cos^2 t_1 + \tilde{p}_2 \cos^2 t_2 + \tilde{p}_3 \cos^2 t_3 & -i\tilde{p}_1 \cos t_1 \sin t_1 & -i\tilde{p}_2 \cos t_2 \sin t_2 & -i\tilde{p}_3 \cos t_3 \sin t_3 \\ i\tilde{p}_1 \cos t_1 \sin t_1 & \tilde{p}_1 \sin^2 t_1 & 0 & 0 \\ i\tilde{p}_2 \cos t_2 \sin t_2 & 0 & \tilde{p}_2 \sin^2 t_2 & 0 \\ i\tilde{p}_3 \cos t_3 \sin t_3 & 0 & 0 & \tilde{p}_3 \sin^2 t_3 \end{bmatrix}.$$

3 Summary

We have shown that QEC without syndrome measurements is possible, such that the output state is a tensor product of a decoded data qubit state and an encoding ancilla state. The recovery operation is combined with the decoding operation, so that both are implemented by a unitary operation. We gave a constructive proof that there always exists such a unitary operator for a given quantum channel. We also prove a result analogous to [2, Theorem 10.2], namely, we show that the recovery operation constructed for a quantum channel Φ in our main theorem is automatically a recovery channel for a channel whose error operators are linear combinations of those of Φ.

Most of the QECs proposed so far are based on the code space. A data qubit state $\tilde{\rho}$ is encoded as ρ and then a noisy quantum channel Φ is applied on ρ. The encoded state is recovered first and subsequently the decoding operation is applied to extract the qubit state $\tilde{\rho}$. Note, however, that what we need eventually is $\tilde{\rho}$ and not ρ. Our QEC is arranged in such a way that the output is a tensor product of $\tilde{\rho}$ and an encoding ancilla state so that $\tilde{\rho}$ is obtained without any syndrome measurement or projection. It follows from our result that QEC can be accomplished by applying a unitary gate followed by a partial trace operation.

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