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Quantum Electrodynamics. III. The Electromagnetic Properties of the Electron —Radiative Corrections to Scattering

JULIAN SCHWINGER
Harvard University, Cambridge, Massachusetts
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The discussion of vacuum polarization in the previous paper of this series was confined to that produced by the field of a prescribed current distribution. We now consider the induction of current in the vacuum by an electron, which is a dynamical system and an entity indistinguishable from the particles associated with vacuum fluctuations. The additional current thus attributed to an electron implies an alteration in its electromagnetic properties which will be revealed by scattering in a Coulomb field and by energy level displacements. This paper is concerned with the computation of the second-order corrections to the current operator and the application to electron scattering. Radiative corrections to energy levels will be treated in the next paper of the series. Following a canonical transformation which effectively renormalizes the electron mass, the correction to the current operator produced by the coupling with the electromagnetic field is developed in a power series, of which first- and second-order terms are retained. One thus obtains second-order modifications in the current operator which are of the same general nature as the previously treated vacuum polarization current, save for a contribution that has the form of a dipole current. The latter implies a fractional increase of $\alpha/2\pi$ in the spin magnetic moment of the electron. The only flaw in the second-order current correction is a logarithmic divergence attributable to an infra-red catastrophe. It is remarked that, in the presence of an external field, the first-order current correction will introduce a compensating divergence. Thus, the second-order corrections to particle electromagnetic properties cannot be completely stated without regard for the manner of exhibiting them by an external field. Accordingly, we consider in the second section the interaction of three systems, the matter field, the electromagnetic field, and a given current distribution. It is shown that this situation can be described in terms of an external potential coupled to the current operator, as modified by the interaction with the vacuum electromagnetic field. Application is made to the scattering of an electron by an external field, in which the latter is regarded as a small perturbation. It is found convenient to calculate the total rate at which collisions occur and then identify the cross sections for individual events. The correction to the cross section for radiationless scattering is determined by the second-order correction to the current operator, while scattering that is accompanied by single quantum emission is a consequence of the first-order current correction. The final object of calculation is the differential cross section for scattering through a given angle with a prescribed maximum energy loss, which is completely free of divergences. Detailed evaluations are given in two situations, the essentially elastic scattering of an electron, in which only a small fraction of the kinetic energy is radiated, and the scattering of a slowly moving electron with unrestricted energy loss. The Appendix is devoted to an alternative treatment of the polarization of the vacuum by an external field. The conditions imposed on the induced current by the charge conservation and gauge invariance requirements are examined. It is found that the fulfillment of these formal properties requires the vanishing of an integral that is not absolutely convergent, but naturally vanishes for reasons of symmetry. This null integral is then used to simplify the expression for the induced current in such a manner that direct calculation yields a gauge invariant result. The induced current contains a logarithmically divergent multiple of the external current, which implies that a non-vanishing total charge, proportional to the external charge, is induced in the vacuum. The apparent contradiction with charge conservation is resolved by showing that a compensating charge escapes to infinity. Finally, the expression for the electromagnetic mass of the electron is treated with the methods developed in this paper.

A COVARIANT form of quantum electrodynamics has been developed, and applied to two elementary vacuum fluctuation phenomena in the previous articles of this series.[1] These applications were the polarization of the vacuum, expressing the modifications in the properties of an electromagnetic field arising from its interaction with the matter field vacuum fluctuations, and the electromagnetic mass of the electron, embodying the corrections to the mechanical properties of the matter field, in its single particle aspect, that are produced by the vacuum fluctuations of the electromagnetic field. In these problems, the divergences that mar the theory are found to be concealed in unobservable charge and mass renormalization factors.

The previous discussion of the polarization of the vacuum was concerned with a given current distribution, one that is not affected by the dynamical reactions of the electron-positron matter field. We shall now consider the more complicated situation in which the original current is that ascribed to an electron or positron—a dynamical system, and an entity indistinguishable from the particles associated with the matter field vacuum fluctuations. The changed electromagnetic properties of the particle will be exhibited in an external field, and may be compared with the experimental indications of deviations from the Dirac theory that were briefly discussed in I. To avoid a work of excessive length, this discussion will be given in two papers. In this paper we shall construct the current operator as modified, to the second order, by the coupling with the vacuum electromagnetic field. This will be applied to compute the radiative correction to the scattering of an electron by a Coulomb field.[2] The second paper will deal with the effects of radiative corrections on energy levels.

I. SECOND-ORDER CORRECTIONS TO THE CURRENT OPERATOR

We shall evaluate the second-order modifications of the current operator produced by the coupling between

[1] Julian Schwinger, "Quantum Electrodynamics. I," Phys. Rev. 74, 1439 (1948); "Quantum Electrodynamics. II," Phys. Rev. 75, 651 (1949).

[2] A short account of the results has already been published, Julian Schwinger, Phys. Rev. 75, 898 (1949).

the matter and electromagnetic fields. The latter is described by

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\mathcal{H}(x)\Psi[\sigma],$$
$$\mathcal{H}(x)=-\frac{1}{c}j_\mu(x)A_\mu(x). \quad (1.1)$$

Among the effects produced by this coupling is the electromagnetic mass of the electron, as contained in the self-energy operator $\mathcal{H}_{1,0}(x)$. In order to describe the electron in terms of the experimental mass, we write (1.1) as

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=(\mathcal{H}_{1,0}(x)+\mathcal{K}(x))\Psi[\sigma], \quad (1.2)$$

where

$$\mathcal{K}(x)=\mathcal{H}(x)-\mathcal{H}_{1,0}(x). \quad (1.3)$$

The canonical transformation

$$\Psi[\sigma]\rightarrow W[\sigma]\Psi[\sigma],$$
$$i\hbar c\frac{\delta W[\sigma]}{\delta\sigma(x)}=\mathcal{H}_{1,0}(x)W[\sigma], \quad (1.4)$$

then replaces (1.2) with

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=W^{-1}[\sigma]\mathcal{K}(x)W[\sigma]\Psi[\sigma], \quad (1.5)$$

while the operator representing the current becomes $W^{-1}[\sigma]j_\mu(x)W[\sigma]$. Now, as we have shown in II, the spinor $W^{-1}[\sigma]\psi(x)W[\sigma]$ obeys the Dirac equation for a particle of mass $m=m_0+\delta m$, the experimental mass of the electron. Accordingly, the expectation value of the current operator can be computed as

$$\langle j_\mu(x)\rangle=(\Psi[\sigma],j_\mu(x)\Psi[\sigma]), \quad (1.6)$$

where

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\mathcal{K}(x)\Psi[\sigma], \quad (1.7)$$

with the understanding that the experimental electron mass is to be employed.

If a solution of the latter equation is constructed in the form

$$\Psi[\sigma]=U[\sigma]\Psi_0, \quad (1.8)$$

the expectation value of the current operator becomes

$$\langle j_\mu(x)\rangle=(\Psi_0,U^{-1}[\sigma]j_\mu(x)U[\sigma]\Psi_0)=(\Psi_0,\mathbf{j}_\mu(x)\Psi_0), \quad (1.9)$$

in which the latter version describes the effect of the coupling between the fields by changing the current operator into

$$\mathbf{j}_\mu(x)=U^{-1}[\sigma]j_\mu(x)U[\sigma]. \quad (1.10)$$

The unitary operator $U[\sigma]$ obeys the equation of motion

$$i\hbar c\frac{\delta U[\sigma]}{\delta\sigma(x)}=\mathcal{K}(x)U[\sigma], \quad (1.11)$$

which may be supplemented by the boundary condition

$$U[-\infty]=1, \quad (1.12)$$

in accordance with the supposition that coupling between the two fields is adiabatically established in the remote past.

The operator $\mathbf{j}_\mu(x)$ can now be evaluated by remarking that

$$\mathbf{j}_\mu(x)=j_\mu(x)+\int_{-\infty}^{\sigma}d\omega'\frac{\delta}{\delta\sigma'(x')}(U^{-1}[\sigma']j_\mu(x)U[\sigma'])$$
$$=j_\mu(x)-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}d\omega' U^{-1}[\sigma'][j_\mu(x),\mathcal{K}(x')]U[\sigma']. \quad (1.13)$$

This process can be continued according to

$$\int_{-\infty}^{\sigma}d\omega' U^{-1}[\sigma'][j_\mu(x),\mathcal{K}(x')]U[\sigma']=\int_{-\infty}^{\sigma}d\omega'[j_\mu(x),\mathcal{K}(x')]$$
$$+\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\frac{\delta}{\delta\sigma''(x'')}(U^{-1}[\sigma''][j_\mu(x),\mathcal{K}(x')]U[\sigma'']), \quad (1.14)$$

and yields $\mathbf{j}_\mu(x)$ in the form of an infinite series,

$$\mathbf{j}_\mu(x)=j_\mu(x)+\left(-\frac{i}{\hbar c}\right)\int_{-\infty}^{\sigma}d\omega'[j_\mu(x),\mathcal{K}(x')]+\left(-\frac{i}{\hbar c}\right)^2\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''[[j_\mu(x),\mathcal{K}(x')],\mathcal{K}(x'')]+\cdots. \quad (1.15)$$

An equivalent procedure, which exhibits $\mathbf{j}_\mu(x)$ in a form that is more symmetrical between past and future, is based on the following observation,

$$\int_{-\infty}^{\infty} d\omega'\epsilon[\sigma,\sigma']\frac{\delta}{\delta\sigma'(x')}(U^{-1}[\sigma']j_\mu(x)U[\sigma'])=\int_{-\infty}^{\sigma} d\omega'\frac{\delta}{\delta\sigma'(x')}(U^{-1}[\sigma']j_\mu(x)U[\sigma'])$$
$$+\int_{\infty}^{\sigma} d\omega'\frac{\delta}{\delta\sigma'(x')}(U^{-1}[\sigma']j_\mu(x)U[\sigma'])=(\mathbf{j}_\mu(x)-j_\mu(x))+(\mathbf{j}_\mu(x)-U^{-1}[\infty]j_\mu(x)U[\infty]), \quad (1.16)$$

or

$$\mathbf{j}_\mu(x)=\tfrac{1}{2}(j_\mu(x)+S^{-1}j_\mu(x)S)+\left(-\frac{i}{2\hbar c}\right)\int_{-\infty}^{\infty} d\omega'\epsilon[\sigma,\sigma']U^{-1}[\sigma'][j_\mu(x),\mathcal{H}(x')]U[\sigma'], \quad (1.17)$$

where

$$S=U[\infty],\quad U[-\infty]=1 \quad (1.18)$$

is the collision operator which describes the real transitions that permanently alter the state of the system. This process can be continued and finally yields

$$\mathbf{j}_\mu(x)=\tfrac{1}{2}(k_\mu(x)+S^{-1}k_\mu(x)S), \quad (1.19)$$

in which

$$k_\mu(x)=j_\mu(x)+\left(-\frac{i}{2\hbar c}\right)\int_{-\infty}^{\infty} d\omega'\epsilon[\sigma,\sigma'][j_\mu(x),\mathcal{H}(x')]$$
$$+\left(-\frac{i}{2\hbar c}\right)^2\int_{-\infty}^{\infty} d\omega' d\omega''\epsilon[\sigma,\sigma']\epsilon[\sigma',\sigma''][[j_\mu(x),\mathcal{H}(x')],\mathcal{H}(x'')]+\cdots. \quad (1.20)$$

The further terms in the series are not required to compute the second-order correction of the current operator.

The collision operator S can be constructed in a similar manner. Thus,

$$S-1=\int_{-\infty}^{\infty} d\omega\frac{\delta}{\delta\sigma(x)}U[\sigma]=-\frac{i}{\hbar c}\int_{-\infty}^{\infty} d\omega\mathcal{H}(x)U[\sigma], \quad (1.21)$$

and

$$U[\sigma]-\tfrac{1}{2}(S+1)=\frac{1}{2}\int_{-\infty}^{\infty} d\omega'\epsilon[\sigma,\sigma']\frac{\delta}{\delta\sigma'(x')}U[\sigma']=-\frac{i}{2\hbar c}\int_{-\infty}^{\infty} d\omega'\epsilon[\sigma,\sigma']\mathcal{H}(x')U[\sigma'], \quad (1.22)$$

whence,

$$S-1=\left(-\frac{i}{2\hbar c}\right)\int_{-\infty}^{\infty} d\omega\mathcal{H}(x)(S+1)+2\left(-\frac{i}{2\hbar c}\right)^2\int_{-\infty}^{\infty} d\omega d\omega'\epsilon[\sigma,\sigma']\mathcal{H}(x)\mathcal{H}(x')U[\sigma']. \quad (1.23)$$

Continuing in this manner, we obtain

$$\frac{S-1}{S+1}=\left(-\frac{i}{2\hbar c}\right)\int_{-\infty}^{\infty} d\omega\mathcal{H}(x)+\left(-\frac{i}{2\hbar c}\right)^2\int_{-\infty}^{\infty} d\omega d\omega'\epsilon[\sigma,\sigma']\mathcal{H}(x)\mathcal{H}(x')+\cdots. \quad (1.24)$$

Only the indicated terms need be retained for the desired degree of approximation. In view of the absence of real first-order effects, as expressed by

$$\int_{-\infty}^{\infty}\mathcal{H}(x)d\omega=0, \quad (1.25)$$

the leading terms in (1.24) are of the second order:

$$\frac{S-1}{S+1}=-\frac{i}{2\hbar c}\int_{-\infty}^{\infty} d\omega\left[-\frac{i}{4\hbar c}\int_{-\infty}^{\infty}\epsilon(x-x')[\mathcal{H}(x),\mathcal{H}(x')]d\omega'-\mathcal{H}_{1,0}(x)\right]. \quad (1.26)$$

According to (II 3.14) and (II 3.71), (the vacuum term $\mathcal{H}_{0,0}$ is of no consequence),

$$-\frac{i}{4\hbar c}\int_{-\infty}^{\infty}\epsilon(x-x')[\mathcal{H}(x),\mathcal{H}(x')]d\omega'=\mathcal{H}_{1,0}(x)+\mathcal{H}_{2,0}(x)+\mathcal{H}_{1,1}(x), \quad (1.27)$$

whence,

$$\frac{S-1}{S+1}=-\frac{i}{2\hbar c}\int_{-\infty}^{\infty}[\mathcal{H}_{2,0}(x)+\mathcal{H}_{1,1}(x)]d\omega, \quad (1.28)$$

which describes the real effects involving either two particles or one particle and a light quantum. Since we shall be concerned only with second-order effects referring to a single particle and the absence of light quanta, such real processes do not come into play and S is effectively unity. Consequently, the current operator is modified only by virtual processes, and is completely symmetrical between past and future. Thus, to the desired order of approximation,

$$\mathfrak{j}_\mu(x)=j_\mu(x)-\frac{i}{2\hbar c}\int_{-\infty}^{\infty}d\omega'\epsilon[\sigma,\sigma'][j_\mu(x),\mathcal{H}(x')]+\frac{i}{2\hbar c}\int_{-\infty}^{\infty}d\omega'\epsilon[\sigma,\sigma'][j_\mu(x),\mathcal{H}_{1,0}(x')]$$
$$+\left(-\frac{i}{2\hbar c}\right)^2\int_{-\infty}^{\infty}d\omega' d\omega''\epsilon[\sigma,\sigma']\epsilon[\sigma',\sigma''][[j_\mu(x),\mathcal{H}(x')],\mathcal{H}(x'')]. \quad (1.29)$$

The correction to the current operator may now be written

$$\mathfrak{j}_\mu(x)-j_\mu(x)=\delta j_\mu^{(1)}(x)+\delta j_\mu^{(2)}(x), \quad (1.30)$$

where

$$\delta j_\mu^{(1)}(x)=\frac{i}{2\hbar c^2}\int_{-\infty}^{\infty}d\omega'\epsilon(x-x')[j_\mu(x),j_\nu(x')]A_\nu(x'), \quad (1.31)$$

and

$$(\delta j_\mu^{(2)}(x))_{1,0}=-\frac{1}{4\hbar^2c^4}\int_{-\infty}^{\infty}d\omega' d\omega''\epsilon[\sigma,\sigma']\epsilon[\sigma',\sigma''][[j_\mu(x),j_\nu(x')]A_\nu(x'),j_\lambda(x'')A_\lambda(x'')]_{1,0}$$
$$+\frac{i}{2\hbar c}\int_{-\infty}^{\infty}d\omega'\epsilon(x-x')[j_\mu(x),\mathcal{H}_{1,0}(x')]_1 \quad (1.32)$$

are the first- and second-order corrections, respectively. In the latter, the subscripts emphasize that we are only concerned with second-order effects involving a single particle and no light quanta. To simplify (1.32), note that

$$[[j_\mu(x),j_\nu(x')]A_\nu(x'),j_\lambda(x'')A_\lambda(x'')]_{1,0}=\tfrac{1}{2}[A_\nu(x'),A_\lambda(x'')]\{[j_\mu(x),j_\nu(x')],j_\lambda(x'')\}_1$$
$$+\tfrac{1}{2}\{A_\nu(x'),A_\lambda(x'')\}_0[[j_\mu(x),j_\nu(x')],j_\lambda(x'')]_1, \quad (1.33)$$

whence

$$(\delta j_\mu^{(2)}(x))_{1,0}=\frac{i}{4\hbar c^3}\int_{-\infty}^{\infty}d\omega' d\omega''\epsilon(x-x')\bar{D}(x'-x'')\{[j_\mu(x),j_\nu(x')],j_\nu(x'')\}_1$$
$$-\frac{1}{8\hbar c^3}\int_{-\infty}^{\infty}d\omega' d\omega''\epsilon[\sigma,\sigma']\epsilon[\sigma',\sigma'']D^{(1)}(x'-x'')[[j_\mu(x),j_\nu(x')],j_\nu(x'')]_1+\frac{i}{2\hbar c}\int_{-\infty}^{\infty}d\omega'\epsilon(x-x')[j_\mu(x),\mathcal{H}_{1,0}(x')]_1, \quad (1.34)$$

in consequence of

$$-\tfrac{1}{2}\epsilon(x'-x'')[A_\nu(x'),A_\lambda(x'')]=i\hbar c\delta_{\nu\lambda}\bar{D}(x'-x'') \quad (1.35)$$

and

$$\{A_\nu(x'),A_\lambda(x'')\}_0=\hbar c\delta_{\nu\lambda}D^{(1)}(x'-x''). \quad (1.36)$$

The double commutator in (1.34) is easily evaluated,

$$[[j_\mu(x),j_\nu(x')],j_\nu(x'')]=-e^2c^2[\bar{\psi}(x)\gamma_\mu S(x-x')\gamma_\nu\psi(x')-\bar{\psi}(x')\gamma_\nu S(x'-x)\gamma_\mu\psi(x),\bar{\psi}(x'')\gamma_\nu\psi(x'')]$$
$$=ie^2c^3(\bar{\psi}(x)\gamma_\mu S(x-x')\gamma_\nu S(x'-x'')\gamma_\nu\psi(x'')+\bar{\psi}(x'')\gamma_\nu S(x''-x')\gamma_\nu S(x'-x)\gamma_\mu\psi(x)$$
$$-\bar{\psi}(x')\gamma_\nu S(x'-x)\gamma_\mu S(x-x'')\gamma_\nu\psi(x'')-\bar{\psi}(x'')\gamma_\nu S(x''-x)\gamma_\mu S(x-x')\gamma_\nu\psi(x')). \quad (1.37)$$

The one-particle part of $\{[j_\mu(x),j_\nu(x')],j_\nu(x'')\}$ can be constructed in the manner employed in II. We have only to notice that $[j_\mu(x),j_\nu(x')]$ has a non-vanishing vacuum expectation value. Thus,

$$\{[j_\mu(x),j_\nu(x')],j_\nu(x'')\}_1-2[j_\mu(x),j_\nu(x')]_0 j_\nu(x'')$$
$$=-e^2c^2\{(\bar{\psi}(x)\gamma_\mu S(x-x')\gamma_\nu\psi(x')-\bar{\psi}(x')\gamma_\nu S(x'-x)\gamma_\mu\psi(x))_1,(\bar{\psi}(x'')\gamma_\nu\psi(x''))_1\}_1$$
$$=-e^2c^3(\bar{\psi}(x)\gamma_\mu S(x-x')\gamma_\nu\{\psi(x'),\bar{\psi}(x'')\}_0\gamma_\nu\psi(x'')-\bar{\psi}(x'')\gamma_\nu\{\psi(x''),\bar{\psi}(x')\}_0\gamma_\nu S(x'-x)\gamma_\mu\psi(x)$$
$$-\bar{\psi}(x')\gamma_\nu S(x'-x)\gamma_\mu\{\psi(x),\bar{\psi}(x'')\}_0\gamma_\nu\psi(x'')+\bar{\psi}(x'')\gamma_\nu\{\psi(x''),\bar{\psi}(x)\}_0\gamma_\mu S(x-x')\gamma_\nu\psi(x'))_1, \quad (1.38)$$

and

$$\{[j_\mu(x),j_\nu(x')],j_\nu(x'')\}_1=2[j_\mu(x),j_\nu(x')]_0 j_\nu(x'')$$
$$+e^3c^3(\bar\psi(x)\gamma_\mu S(x-x')\gamma_\nu S^{(1)}(x'-x'')\gamma_\nu\psi(x'')-\bar\psi(x'')\gamma_\nu S^{(1)}(x''-x')\gamma_\nu S(x'-x)\gamma_\mu\psi(x)$$
$$-\bar\psi(x')\gamma_\nu S(x'-x)\gamma_\mu S^{(1)}(x-x'')\gamma_\nu\psi(x'')+\bar\psi(x'')\gamma_\nu S^{(1)}(x''-x)\gamma_\mu S(x-x')\gamma_\nu\psi(x'))_1. \quad (1.39)$$

On inserting (1.37) and (1.39) into (1.34), we obtain

$$(\delta j_\mu{}^{(2)}(x))_{1,0}=\frac{i}{2\hbar c^2}\int_{-\infty}^{\infty}d\omega' d\omega''\epsilon(x-x')[j_\mu(x),j_\nu(x')]_0\bar D(x'-x'')j_\nu(x'')$$
$$-\frac{ie^3}{2\hbar}\int_{-\infty}^{\infty}d\omega' d\omega''(\bar\psi(x')\gamma_\nu\bar S(x'-x)\gamma_\mu S^{(1)}(x-x'')\gamma_\nu\psi(x'')+\bar\psi(x')\gamma_\nu S^{(1)}(x'-x)\gamma_\mu\bar S(x-x'')\gamma_\nu\psi(x''))_1\bar D(x'-x'')$$
$$-\frac{ie^3}{2\hbar}\int_{-\infty}^{\infty}d\omega' d\omega''(\bar\psi(x')\gamma_\nu\bar S(x'-x)\gamma_\mu\bar S(x-x'')\gamma_\nu\psi(x''))_1 D^{(1)}(x'-x'')$$
$$+\frac{ie}{\hbar}\int_{-\infty}^{\infty}d\omega'(\bar\psi(x)\gamma_\mu\bar S(x-x')\phi(x')+\bar\phi(x')\bar S(x'-x)\gamma_\mu\psi(x))_1$$
$$-\frac{e}{2\hbar}\int_{-\infty}^{\infty}d\omega'\epsilon(x-x')(\bar\psi(x)\gamma_\mu[\psi(x),\mathcal{H}_{1,0}(x')]+[\bar\psi(x),\mathcal{H}_{1,0}(x')]\gamma_\mu\psi(x))_1, \quad (1.40)$$

where (see II (3.78))

$$\phi(x)=-\frac{e^2}{2}\int_{-\infty}^{\infty}d\omega'\gamma_\nu(\bar D(x-x')S^{(1)}(x-x')+D^{(1)}(x-x')\bar S(x-x'))\gamma_\nu\psi(x')$$
$$=\delta mc^2\psi(x). \quad (1.41)$$

The third term of (1.40) is derived from

$$\frac{ie^3}{8\hbar}\int_{-\infty}^{\infty}d\omega' d\omega''(\epsilon[\sigma,\sigma']-\epsilon[\sigma,\sigma''])\epsilon[\sigma',\sigma''](\bar\psi(x')\gamma_\nu S(x'-x)\gamma_\mu S(x-x'')\gamma_\nu\psi(x''))_1 D^{(1)}(x'-x'') \quad (1.42)$$

with the aid of the identity

$$(\epsilon[\sigma,\sigma']-\epsilon[\sigma,\sigma''])\epsilon[\sigma',\sigma'']=\epsilon[\sigma,\sigma']\epsilon[\sigma,\sigma'']-1, \quad (1.43)$$

and the null value of

$$\int_{-\infty}^{\infty}d\omega' d\omega''(\bar\psi(x')\gamma_\nu S(x'-x)\gamma_\mu S(x-x'')\gamma_\nu\psi(x''))_1 D^{(1)}(x'-x'')$$
$$=-\frac{1}{\hbar c}\int_{-\infty}^{\infty}d\omega' d\omega''(\bar\psi(x')\gamma_\nu\{\psi(x'),\bar\psi(x)\}\gamma_\mu\{\psi(x),\bar\psi(x'')\}\gamma_\lambda\psi(x''))_1\{A_\nu(x'),A_\lambda(x'')\}_0. \quad (1.44)$$

The latter is an immediate consequence of (1.25), expressing the absence of real first-order transitions.

The insertion of the expression for $\mathcal{H}_{1,0}(x)$, (II (3.77)),

$$\mathcal{H}_{1,0}(x)=\tfrac{1}{2}(\bar\psi(x)\phi(x)+\bar\phi(x)\psi(x))_1, \quad (1.45)$$

enables the last two terms of (1.40) to be combined into

$$\frac{ie}{2\hbar}(\bar\psi(x)\gamma_\mu\chi(x)+\bar\chi(x)\gamma_\mu\psi(x))_1, \quad (1.46)$$

where

$$\chi(x)=\int_{-\infty}^{\infty}d\omega'\left[\bar S(x-x')\phi(x')+\frac{i}{2}\epsilon(x-x')\{\psi(x),\bar\phi(x')\}\psi(x')\right]. \quad (1.47)$$

It will first be observed that the integrand of (1.47) vanishes, in virtue of the relation $\phi(x)=\delta mc^2\psi(x)$, since

$$\frac{i}{2}\epsilon(x-x')\{\psi(x),\bar\phi(x')\}\psi(x')=-\bar S(x-x')\delta mc^2\psi(x'). \quad (1.48)$$

On the other hand, integrals of the form

$$\int_{-\infty}^{\infty} d\omega' \bar{S}(x-x')\psi(x') \tag{1.49}$$

are divergent, since $\psi(x')$ and $\bar{S}(x-x')$, respectively, obey homogeneous and inhomogeneous equations associated with the same differential operator. It is convenient to express the latter integral as the limit of the finite quantity obtained by an alteration of the differential equation satisfied by $\psi(x)$, in which the mass parameter κ is replaced by $\kappa+\delta\kappa$ and the limit $\delta\kappa\to 0$ is taken. The differential equations

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu'}+\kappa+\delta\kappa\right)\psi(x')=0, \quad \frac{\partial}{\partial x_\mu'}\bar{S}(x-x')\gamma_\mu-\kappa\bar{S}(x-x')=\delta(x-x'), \tag{1.50}$$

imply the relation

$$\frac{\partial}{\partial x_\mu'}[\bar{S}(x-x')\gamma_\mu\psi(x')]+\delta\kappa\bar{S}(x-x')\psi(x')=\delta(x-x')\psi(x), \tag{1.51}$$

whence

$$\int_{-\infty}^{\infty} d\omega' \bar{S}(x-x')\psi(x')=\lim_{\delta\kappa\to 0}\frac{1}{\delta\kappa}\psi(x). \tag{1.52}$$

It may be inferred that a non-vanishing value will be obtained for $\chi(x)$ if the spinor, of which (1.47) is a linear function, is subject to the Dirac equation with the mass parameter $\kappa+\delta\kappa$, and $\delta\kappa$ is allowed to approach zero. Indeed, according to (1.48) and (1.52),

$$\chi(x)=\lim_{\delta\kappa\to 0}\int_{-\infty}^{\infty} d\omega' \bar{S}(x-x')(\phi(x')-\delta mc^2\psi(x'))$$

$$=\lim_{\delta\kappa\to 0}\frac{1}{\delta\kappa}(\phi(x)-\delta mc^2\psi(x)), \tag{1.53}$$

or

$$\chi(x)=\lim_{\delta\kappa\to 0}\frac{1}{\delta\kappa}\left[-\frac{e^2}{2}\int_{-\infty}^{\infty} d\omega'\gamma_\nu(\bar{D}(x-x')S^{(1)}(x-x')+D^{(1)}(x-x')\bar{S}(x-x'))\gamma_\nu\psi(x')-\delta mc^2\psi(x)\right]. \tag{1.54}$$

A suitable representation of the solution of the Dirac equation with an altered mass parameter, $\psi_{\kappa+\delta\kappa}(x')$, in terms of the actual spinor, $\psi_\kappa(x')$, is provided by

$$\psi_{\kappa+\delta\kappa}(x')=\psi_\kappa(x')+\frac{\delta\kappa}{\kappa}(x_\lambda'-x_\lambda)\frac{\partial}{\partial x_\lambda'}\psi_\kappa(x') \tag{1.55}$$

since

$$\left(\gamma_\mu\frac{\partial}{\partial x_\mu'}+\kappa\right)\psi_{\kappa+\delta\kappa}(x')=\frac{\delta\kappa}{\kappa}\gamma_\mu\frac{\partial}{\partial x_\mu'}\psi_\kappa(x')$$

$$=-\delta\kappa\psi_\kappa(x'), \tag{1.56}$$

which establishes the validity of (1.55) to the first order in $\delta\kappa$. The latter is so constructed that $\psi_{\kappa+\delta\kappa}(x)=\psi_\kappa(x)$, whence,

$$\chi(x)=\frac{e^2}{2\kappa}\int_{-\infty}^{\infty} d\omega'\gamma_\nu(\bar{D}(x-x')S^{(1)}(x-x')+D^{(1)}(x-x')\bar{S}(x-x'))\gamma_\nu(x_\lambda-x_\lambda')\frac{\partial}{\partial x_\lambda'}\psi(x'). \tag{1.57}$$

The resulting expression for the second-order correction to the current operator is

$$(\delta j_\mu^{(2)}(x))_{1,0}-\frac{i}{2\hbar c^2}\int_{-\infty}^{\infty} d\omega'\epsilon(x-x')[j_\mu(x),j_\nu(x')]_0\delta A_\nu(x')-\frac{ie^3}{2\hbar}\int_{-\infty}^{\infty} d\omega' d\omega''(\bar{\psi}(x')K_\mu(x'-x,x-x'')\psi(x''))_1, \tag{1.58}$$

where

$$\delta A_\mu(x)=\frac{1}{c}\int_{-\infty}^{\infty} d\omega'\bar{D}(x-x')j_\mu(x'), \tag{1.59}$$

and

$$K_\mu(x'-x,x-x'')=K_\mu^{(1)}(x'-x,x-x'')+K_\mu^{(2)}(x'-x,x-x''). \tag{1.60}$$

Here

$$K_\mu^{(1)}(\xi,\eta)=\gamma_\nu(\bar{S}(\xi)\gamma_\mu S^{(1)}(\eta)\bar{D}(\xi+\eta)+S^{(1)}(\xi)\gamma_\mu\bar{S}(\eta)\bar{D}(\xi+\eta)+\bar{S}(\xi)\gamma_\mu\bar{S}(\eta)D^{(1)}(\xi+\eta))\gamma_\nu \tag{1.61}$$

and

$$K_\mu^{(2)}(\xi,\eta)=-\gamma_\mu\delta(\xi)\frac{1}{2\kappa}\frac{\partial}{\partial\eta_\lambda}\eta_\lambda\gamma_\nu(\bar{D}(\eta)S^{(1)}(\eta)+D^{(1)}(\eta)\bar{S}(\eta))\gamma_\nu-\frac{1}{2\kappa}\frac{\partial}{\partial\xi_\lambda}\xi_\lambda\gamma_\nu(\bar{D}(\xi)S^{(1)}(\xi)+D^{(1)}(\xi)\bar{S}(\xi))\gamma_\nu\delta(\eta)\gamma_\mu. \quad (1.62)$$

An equivalent form can be given in terms of the functions

$$S_\pm(x)=\bar{S}(x)\pm\frac{i}{2}S^{(1)}(x),$$

$$D_\pm(x)=\bar{D}(x)\pm\frac{i}{2}D^{(1)}(x). \quad (1.63)$$

Thus,

$$K_\mu^{(1)}(\xi,\eta)=\frac{1}{i}\gamma_\nu(S_+(\xi)\gamma_\mu S_+(\eta)D_+(\xi+\eta)-S_-(\xi)\gamma_\mu S_-(\eta)D_-(\xi+\eta))\gamma_\nu, \quad (1.64)$$

and

$$K_\mu^{(2)}(\xi,\eta)=-\gamma_\mu\delta(\xi)\frac{1}{2\kappa}\frac{\partial}{\partial\eta_\lambda}\eta_\lambda\frac{1}{i}\gamma_\nu(D_+(\eta)S_+(\eta)-D_-(\eta)S_-(\eta))\gamma_\nu$$
$$-\frac{1}{2\kappa}\frac{\partial}{\partial\xi_\lambda}\xi_\lambda\frac{1}{i}\gamma_\nu(D_+(\xi)S_+(\xi)-D_-(\xi)S_-(\xi))\gamma_\nu\delta(\eta)\gamma_\mu. \quad (1.65)$$

The first term of (1.58) is the current induced by the electromagnetic field that accompanies a given current distribution, as discussed in II. It is the second part of (1.58), expressing the additional effects involved when the current is associated with the matter field, rather than an external system, that merits our attention.

In order to evaluate $K_\mu(x'-x,x-x'')$, we shall substitute Fourier integral representations for the various functions involved, (II (A.10), (A.31)),

$$\bar{S}(x)=\frac{1}{(2\pi)^4}\int(dk)e^{ikx}(i\gamma k-\kappa)\frac{1}{k^2+\kappa^2},$$

$$S^{(1)}(x)=\frac{1}{(2\pi)^3}\int(dk)e^{ikx}(i\gamma k-\kappa)\delta(k^2+\kappa^2),$$

$$\bar{D}(x)=\frac{1}{(2\pi)^4}\int(dk)e^{ikx}\frac{1}{k^2},$$

$$D^{(1)}(x)=\frac{1}{(2\pi)^3}\int(dk)e^{ikx}\delta(k^2), \quad (1.66)$$

in which the principal part of $1/(k^2+\kappa^2)$ and $1/k^2$ is understood. We have employed the simplified notation ab to denote $a_\mu b_\mu$, the scalar product of two four-vectors. The functions (1.63) have the following Fourier integral representations,

$$S_\pm(x)=\frac{1}{(2\pi)^4}\int(dk)e^{ikx}(i\gamma k-\kappa)\left(\frac{1}{k^2+\kappa^2}\pm\pi i\delta(k^2+\kappa^2)\right),$$

$$D_\pm(x)=\frac{1}{(2\pi)^4}\int(dk)e^{ikx}\left(\frac{1}{k^2}\pm\pi i\delta(k^2)\right). \quad (1.67)$$

These expressions can be written more compactly by observing that

$$\lim_{\epsilon\to+0}\frac{1}{\xi\mp i\epsilon}=\lim_{\epsilon\to+0}\left(\frac{\xi}{\xi^2+\epsilon^2}\pm i\frac{\epsilon}{\xi^2+\epsilon^2}\right)$$
$$=P\frac{1}{\xi}\pm\pi i\delta(\xi), \quad (1.68)$$

whence,

$$S_\pm(x)=\frac{1}{(2\pi)^4}\int(dk)e^{ikx}(i\gamma k-\kappa)\frac{1}{k^2+\kappa^2\mp i\epsilon},$$

$$D_\pm(x)=\frac{1}{(2\pi)^4}\int(dk)e^{ikx}\frac{1}{k^2\mp i\epsilon},\tag{1.69}$$

in which the limit $\epsilon\to+0$ is understood.

The form obtained for $K_\mu^{(1)}$ from (1.61), is

$$K_\mu^{(1)}(x'-x,x-x'')=\frac{1}{(2\pi)^{11}}\int(dk)(dk')(dk'')e^{i(k+k')(x'-x)}e^{i(k+k'')(x-x'')}$$
$$\times\gamma_\nu(i\gamma k'-\kappa)\gamma_\mu(i\gamma k''-\kappa)\gamma_\nu\left[\frac{\delta(k'^2+\kappa^2)}{(k''^2+\kappa^2)k^2}+\frac{\delta(k''^2+\kappa^2)}{(k'^2+\kappa^2)k^2}+\frac{\delta(k^2)}{(k'^2+\kappa^2)(k''^2+\kappa^2)}\right].\tag{1.70}$$

It is convenient to replace k_μ' and k_μ'' by

$$p_\mu'=k_\mu+k_\mu',\quad p_\mu''=k_\mu+k_\mu'',\tag{1.71}$$

which enter directly in the coordinate dependence of the Fourier integral. Since $K_\mu^{(1)}(x'-x,x-x'')$ is to be multiplied by $\bar\psi(x')$, $\psi(x'')$ and integrated with respect to x' and x'', only such values of p_μ' and p_μ'' occur for which

$$p'^2+\kappa^2=p''^2+\kappa^2=0.\tag{1.72}$$

As a result of this transformation,

$$K_\mu^{(1)}(x'-x,x-x'')=\frac{1}{(2\pi)^{11}}\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}\gamma_\nu(i\gamma(p'-k)-\kappa)\gamma_\mu(i\gamma(p''-k)-\kappa)\gamma_\nu$$
$$\times\left[\frac{\delta(k^2-2kp')}{(2kp'-2kp'')(2kp')}+\frac{\delta(k^2-2kp'')}{(2kp''-2kp')(2kp'')}+\frac{\delta(k^2)}{(2kp')(2kp'')}\right].\tag{1.73}$$

The last factor in (1.73) can be simplified by writing it as

$$\frac{1}{2k(p'-p'')}\left[\frac{1}{2kp'}(\delta(k^2-2kp')-\delta(k^2))-\frac{1}{2kp''}(\delta(k^2-2kp'')-\delta(k^2))\right],\tag{1.74}$$

and observing that

$$\frac{1}{2kp}(\delta(k^2-2kp)-\delta(k^2))=-\int_0^1du\delta'(k^2-2kpu),\tag{1.75}$$

whence (1.74) becomes

$$-\frac{1}{2k(p'-p'')}\int_0^1du[\delta'(k^2-2kp'u)-\delta'(k^2-2kp''u)].\tag{1.76}$$

This, in turn, can be represented more compactly as

$$\frac{1}{2}\int_{-1}^1dv\int_0^1udu\delta''(k^2-k(p'+p''+(p'-p'')v)u).\tag{1.77}$$

Therefore,

$$K_\mu^{(1)}(x'-x,x-x'')=\frac{1}{2(2\pi)^{11}}\int_{-1}^1dv\int_0^1udu\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}$$
$$\times\gamma_\nu(i\gamma(p'-k)-\kappa)\gamma_\mu(i\gamma(p''-k)-\kappa)\gamma_\nu\delta''(k^2-k(p'+p''+(p'-p'')v)u).\tag{1.78}$$

If the expression (1.64) is employed for $K_\mu^{(1)}$, the bracketed factors in (1.70) and (1.73) are replaced by

$$\frac{1}{\pi}Im\frac{1}{k'^2+\kappa^2-i\epsilon}\,\frac{1}{k''^2+\kappa^2-i\epsilon}\,\frac{1}{k^2-i\epsilon}=\frac{1}{\pi}Im\frac{1}{k^2-2kp'-i\epsilon}\,\frac{1}{k^2-2kp''-i\epsilon}\,\frac{1}{k^2-i\epsilon}.\tag{1.79}$$

However,

$$\frac{1}{k^2-2kp'-i\epsilon}\frac{1}{k^2-2kp''-i\epsilon}\frac{1}{k^2-i\epsilon}=\frac{1}{2k(p'-p'')}\left[\frac{1}{2kp'}\left(\frac{1}{k^2-2kp'-i\epsilon}-\frac{1}{k^2-i\epsilon}\right)-\frac{1}{2kp''}\left(\frac{1}{k^2-2kp''-i\epsilon}-\frac{1}{k^2-i\epsilon}\right)\right], \quad (1.80)$$

and, on extracting the imaginary part divided by π, we again encounter (1.74).

The second part of K_μ, (1.62), can also be readily expressed in Fourier integral form

$$K_\mu^{(2)}(x'-x,x-x'')=\frac{1}{(2\pi)^{11}}\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}\left[\frac{1}{2\kappa}p_\lambda'\frac{\partial}{\partial p_\lambda'}\gamma_\nu(i\gamma(p'-k)-\kappa)\gamma_\nu\left(\frac{\delta((k-p')^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{(k-p')^2+\kappa^2}\right)\gamma_\mu+\gamma_\mu\frac{1}{2\kappa}p_\lambda''\frac{\delta}{\partial p_\lambda''}\gamma_\nu(i\gamma(p''-k)-\kappa)\gamma_\nu\left(\frac{\delta((k-p'')^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{(k-p'')^2+\kappa^2}\right)\right]. \quad (1.81)$$

To evaluate the derivatives with respect to p_λ' and p_λ'', we observe that

$$p_\lambda\frac{\partial}{\partial p_\lambda}(i\gamma(p-k)+\kappa)(i\gamma(p-k)-\kappa)f((p-k)^2+\kappa^2)=0, \quad (1.82)$$

where $f(x)$ is $\delta(x)$ or $1/x$. On differentiating and multiplying to the left by $i\gamma(p-k)-\kappa$, we obtain

$$p_\lambda\frac{\partial}{\partial p_\lambda}(i\gamma(p-k)-\kappa)f((p-k)^2+\kappa^2)=(i\gamma(p-k)-\kappa)i\gamma p(i\gamma(p-k)-\kappa)\frac{f(k^2-2kp)}{k^2-2kp}. \quad (1.83)$$

Consequently,

$$p_\lambda\frac{\partial}{\partial p_\lambda}\gamma_\nu(i\gamma(p-k)-\kappa)\gamma_\nu\left(\frac{\delta((p-k)^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{(p-k)^2+\kappa^2}\right)=-\gamma_\nu(i\gamma(p-k)-\kappa)i\gamma p(i\gamma(p-k)-\kappa)\gamma_\nu\left(\frac{\delta'(k^2-2kp)}{k^2}-\frac{(\delta k^2)}{(2kp)^2}\right), \quad (1.84)$$

in virtue of the delta-function property

$$\delta'(x)=-\frac{\delta(x)}{x}. \quad (1.85)$$

Furthermore,

$$\frac{\delta'(k^2-2kp)}{k^2}-\frac{\delta(k^2)}{(2kp)^2}=-\frac{\partial}{\partial(2kp)}\left(\frac{\delta(k^2-2kp)}{k^2}-\frac{\delta(k^2)}{2kp}\right)=-\int_0^1 udu\delta''(k^2-2kpu), \quad (1.86)$$

according to (1.75). Therefore, (1.81) becomes

$$K_\mu^{(2)}(x'-x,x-x'')=\frac{1}{(2\pi)^{11}}\int_0^1 udu\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}\times\left[\delta''(k^2-2kp'u)\frac{1}{2\kappa}\gamma_\nu(i\gamma(p'-k)-\kappa)i\gamma p'(i\gamma(p'-k)-\kappa)\gamma_\nu\gamma_\mu+\gamma_\mu\frac{1}{2\kappa}\gamma_\nu(i\gamma(p'-k)-\kappa)i\gamma p''(i\gamma(p''-k)-\kappa)\gamma_\nu\delta''(k^2-2kp''u)\right]. \quad (1.87)$$

The transformation

$$k_\mu\rightarrow k_\mu+(p_\mu'+p_\mu''+(p_\mu'-p_\mu'')v)\frac{u}{2} \quad (1.88)$$

now brings the delta-function of (1.78) into the form

$$\delta''(k^2+\lambda^2u^2), \quad (1.89)$$

where

$$\lambda^2=\kappa^2\left(1+\frac{(p'-p'')^2}{4\kappa^2}(1-v^2)\right), \tag{1.90}$$

in virtue of the relations

$$\left(\frac{p'+p''}{2}\right)^2+\left(\frac{p'-p''}{2}\right)^2+\kappa^2=0,\quad (p'+p'')(p'-p'')=0. \tag{1.91}$$

As a consequence of this transformation, the factor involving the Dirac matrices in (1.78) becomes

$$\gamma_\nu\left(i\gamma\left(p'-\frac{p'+p''}{2}u-\frac{p'-p''}{2}uv\right)-\kappa\right)\gamma_\mu\left(i\gamma\left(p''-\frac{p'+p''}{2}u-\frac{p'-p''}{2}uv\right)-\kappa\right)\gamma_\nu-\gamma_\mu k^2. \tag{1.92}$$

In writing this result we have exploited the symmetry of the delta-function (1.89), in connection with the k integration, and discarded terms linear in k_λ, while replacing $k_\lambda k_\nu$ by $\frac{1}{4}\delta_{\lambda\nu}k^2$. The following property of the Dirac matrices has also been used,

$$\gamma_\lambda\gamma_\mu\gamma_\lambda=-2\gamma_\mu. \tag{1.93}$$

The factor (1.92) can be further simplified by omitting the terms linear in v, which will vanish on integration, and rearranging the remaining terms to obtain

$$\begin{aligned}&4\kappa^2\gamma_\mu(1-u-\tfrac{1}{2}u^2)-\gamma_\mu k^2+2\kappa(u-u^2)\sigma_{\mu\nu}(p_\nu'-p_\nu'')+2(p'-p'')^2\gamma_\mu\left(1-u+\frac{1-v^2}{4}u^2\right)\\&\quad+i(1-u^2v^2)(p_\mu'-p_\mu'')((i\gamma p'+\kappa)-(i\gamma p''+\kappa))-2(1-u)\bigg[(i\gamma p'+\kappa)\left(\kappa(1+u)\gamma_\mu+ip_\mu''+i\frac{1-u}{2}(p_\mu'+p_\mu'')\right)\\&\quad-(i\gamma p'+\kappa)\gamma_\mu(i\gamma p''+\kappa)+\left(\kappa(1+u)\gamma_\mu+ip_\mu'+i\frac{1-u}{2}(p_\mu'+p_\mu'')\right)(i\gamma p''+\kappa)\bigg].\end{aligned} \tag{1.94}$$

Now, a right-hand factor $i\gamma p''+\kappa$ is equivalent to $-\gamma_\lambda(\partial/\partial x_\lambda'')+\kappa$ operating on $K_\mu(x'-x,x-x'')$, which annihilates $\psi(x'')$ on integration by parts. Similarly, a left-hand factor $i\gamma p'+\kappa$ annihilates $\bar{\psi}(x')$. As a consequence of the Dirac equation, therefore,

$$\begin{aligned}K_\mu^{(1)}(x'-x,x-x'')=&\frac{1}{(2\pi)^{11}}\int_{-1}^{1}dv\int_0^1 udu\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}\delta''(k^2+\lambda^2u^2)\\&\times\left[2\kappa^2\gamma_\mu(1-u-\tfrac{1}{2}u^2)-\tfrac{1}{2}\gamma_\mu k^2+\kappa(u-u^2)\sigma_{\mu\nu}(p_\nu'-p_\nu'')+(p'-p'')^2\gamma_\mu\left(1-u+\frac{1-v^2}{4}u^2\right)\right].\end{aligned} \tag{1.95}$$

Transformations analogous to (1.88) can be introduced in the two terms of (1.87), namely $k_\mu\rightarrow k_\mu+p_\mu'u$, and $k_\mu\rightarrow k_\mu+p_\mu''u$. Both delta functions then become $\delta''(k^2+\kappa^2u^2)$, while the factors involving the Dirac matrices simplify according to

$$\frac{1}{2\kappa}\gamma_\nu(i\gamma(p-k)-\kappa)i\gamma p(i\gamma(p-k)-\kappa)\gamma_\nu\rightarrow-2\kappa^2(1-u-\tfrac{1}{2}u^2)+\tfrac{1}{2}k^2-(2\kappa^2(1-u+\tfrac{1}{2}u^2)+\tfrac{1}{2}k^2)\frac{1}{\kappa}(i\gamma p+\kappa), \tag{1.96}$$

where p is p' or p'' for the two terms of (1.87). In consequence of the Dirac equation, therefore,

$$\begin{aligned}K_\mu^{(2)}(x'-x,x-x'')=-\frac{2}{(2\pi)^{11}}\int_0^1 udu\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{p''(x-x'')}\delta''(k^2+\kappa^2u^2)&\\\times[2\kappa^2\gamma_\mu(1-u-\tfrac{1}{2}u^2)-\tfrac{1}{2}\gamma_\mu k^2].&\end{aligned} \tag{1.97}$$

To combine $K_\mu^{(1)}$ and $K_\mu^{(2)}$, it is sufficient to perform an integration by parts with respect to v for the first two terms of (1.95), as indicated by

$$\int_{-1}^{1}dv\delta''(k^2+\lambda^2u^2)=2\delta''(k^2+\kappa^2u^2)-\int_1^1 dvv\frac{\partial}{\partial v}\delta''(k^2+\lambda^2u^2). \tag{1.98}$$

The integrated terms precisely cancel $K_\mu^{(2)}$. If the v differentiation is explicitly performed for the second term of (1.95), the k integral thus encountered is

$$\int (dk)k^2\delta'''(k^2+\lambda^2u^2)=\frac{1}{2}\int (dk)k_\nu\frac{\partial}{\partial k_\nu}\delta''(k^2+\lambda^2u^2)$$

$$=-2\int (dk)\delta''(k^2+\lambda^2u^2). \tag{1.99}$$

Hence,

$$K_\mu(x'-x,x-x'')=\frac{1}{(2\pi)^{11}}\int_{-1}^{1}dv\int_0^1 udu\int (dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}\Big[(p'-p'')^2\gamma_\mu\Big(1-u+\frac{1+v^2}{4}u^2\Big)$$

$$+\kappa(u-u^2)\sigma_{\mu\nu}(p_\nu'-p_\nu'')-2\kappa^2\gamma_\mu(1-u-\tfrac{1}{2}u^2)v\frac{\partial}{\partial v}\Big]\delta''(k^2+\lambda^2u^2). \tag{1.100}$$

The integration with respect to k may now be effected. According to the integral representation,

$$\delta(k^2+\lambda^2u^2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dwe^{iw(k^2+\lambda^2u^2)}, \tag{1.101}$$

we have

$$\int (dk)\delta''(k^2+\lambda^2u^2)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}w^2dwe^{iw\lambda^2u^2}\int (dk)e^{iwk^2}$$

$$=-\frac{\pi i}{2}\int_{-\infty}^{\infty}dw\frac{w}{|w|}e^{iw\lambda^2u^2}$$

$$=\frac{\pi}{\lambda^2u^2}. \tag{1.102}$$

However, it should be noticed that we are then required to evaluate integrals with respect to u of the form

$$\int_0^1 u^{n+1}du\int (dk)\delta''(k^2+\lambda^2u^2)=\frac{\pi}{\lambda^2}\int_0^1 u^{n-1}du, \tag{1.103}$$

in which n may be 0, 1 or 2. For $n=0$, the integral is logarithmically divergent.

In order to ascertain the significance of this divergence, we shall interchange the operations used in obtaining (1.103), thus producing a more easily interpreted divergent k integral. For $n=0$, (1.103) reads

$$\frac{1}{2\lambda^2}\int (dk)\int_0^1 du\frac{\partial}{\partial u}\delta'(k^2+\lambda^2u^2)=\frac{1}{2\lambda^2}\int (dk)[\delta'(k^2+\lambda^2)-\delta'(k^2)]. \tag{1.104}$$

One may express this invariant integral, in three-dimensional notation, as

$$\frac{1}{2\lambda^2}\int (d\mathbf{k})dk_0\frac{1}{2k_0}\frac{\partial}{\partial k_0}[\delta(k_0^2-\mathbf{k}^2)-\delta(k_0^2-\mathbf{k}^2-\lambda^2)]=\frac{1}{4\lambda^2}\int (d\mathbf{k})dk_0\frac{1}{k_0^2}[\delta(k_0^2-\mathbf{k}^2)-\delta(k_0^2-\mathbf{k}^2-\lambda^2)], \tag{1.105}$$

in which the delta-functions describe the energy-momentum relations of a light quantum, and of a particle with mass $\hbar\lambda/c$. On performing the k_0 integration, (1.105) becomes

$$\frac{1}{4\lambda^2}\int (d\mathbf{k})\Big[\frac{1}{|\mathbf{k}|^3}-\frac{1}{(\mathbf{k}^2+\lambda^2)^{\frac{3}{2}}}\Big]=\frac{\pi}{\lambda^2}\Big[\int_0^\infty dk\Big(\frac{1}{k}-\frac{1}{(k^2+\lambda^2)^{\frac{1}{2}}}\Big)+1\Big], \tag{1.106}$$

in which form it is evident that the divergence is associated with zero frequency light quanta—an "infra-red catastrophe." As we shall later demonstrate, this divergence is entirely spurious, and is removed on properly including the effects of $\delta j_\mu^{(1)}(x)$, the first-order correction to the current operator. The divergent integral (1.106) can be expressed in terms of an invariant minimum light quantum wave number, $k_{\min}$, as

$$\frac{\pi}{\lambda^2}\left(\log\frac{\lambda}{2k_{\min}}+1\right). \tag{1.107}$$

With the k and u integrations thus performed, $K_\mu(x'-x,x-x'')$ becomes

$$K_\mu(x'-x,x-x'')=\frac{1}{(2\pi)^{10}}\int_0^1 dv\int(dp')(dp'')\exp\left[i\frac{p'+p''}{2}(x'-x'')\right]\exp\left[i(p''-p')\left(x-\frac{x'+x''}{2}\right)\right]$$
$$\times\left[\frac{(p'-p'')^2}{\kappa^2}\gamma_\mu\frac{1+v^2}{2}\left(\log\frac{\kappa}{2k_{\min}}+1+\tfrac{1}{2}\log\left(1+\frac{(p'-p'')^2}{4\kappa^2}(1-v^2)\right)\right)-\frac{(p'-p'')^2}{4\kappa^2}\gamma_\mu\right.$$
$$\left.+\frac{1}{2\kappa}\sigma_{\mu\nu}(p_\nu'-p_\nu'')\right]\frac{1}{1+((p'-p'')^2/4\kappa^2)(1-v^2)}, \tag{1.108}$$

in which we have evaluated the third term of (1.100) by writing

$$\frac{1}{1+((p'-p'')^2/4\kappa^2)(1-v^2)}=1-\frac{((p'-p'')^2/4\kappa^2)(1-v^2)}{1+((p'-p'')^2/4\kappa^2)(1-v^2)}, \tag{1.109}$$

and performing the v differentiation for the term obtained from the first part of (1.109), while reversing the integration by parts for the term produced by the second part of (1.109). It will now be observed that the integrand of (1.108) involves only $p_\lambda'-p_\lambda''$. It is then useful to introduce the new variables

$$P_\lambda=\frac{p_\lambda'+p_\lambda''}{2},\quad p_\lambda=p_\lambda''-p_\lambda', \tag{1.110}$$

since the P integration can be immediately performed, yielding $\delta(x'-x'')$. In this way, we obtain

$$K_\mu(x'-x,x-x'')=-\frac{1}{8\pi^2}\gamma_\mu\delta(x'-x'')\frac{1}{\kappa^2}\Box^2\left[\log\frac{\kappa}{2k_{\min}}(F_0(x-x')+F_1(x-x'))\right.$$
$$\left.+\tfrac{1}{2}F_0(x-x')+F_1(x-x')+\tfrac{1}{2}G(x-x')\right]+\frac{i}{8\pi^2}\delta(x'-x'')\sigma_{\mu\nu}\frac{\partial}{\partial x_\nu}F_0(x-x'), \tag{1.111}$$

where

$$F_n(x)=\int_0^1 dv v^{2n}\frac{1}{(2\pi)^4}\int(dp)\frac{e^{ipx}}{1+(p^2/4k^2)(1-v^2)}$$
$$=16\kappa^2\int_0^1 dv\frac{v^{2n}}{(1-v^2)^2}\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}x\right) \tag{1.112}$$

and

$$G(x)=\int_0^1 dv(1+v^2)\frac{1}{(2\pi)^4}\int(dp)e^{ipx}\frac{\log(1+(p^2/4\kappa^2)(1-v^2))}{1+(p^2/4\kappa^2)(1-v^2)}$$
$$=8\Box^2\int_0^1 dv\frac{1+v^2}{1-v^2}\int_0^1\frac{udu}{1-u^2}\left[\frac{1}{u^2}\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}u}x\right)-\bar{\Delta}\left(\frac{2}{(1-v^2)^{\frac{1}{2}}}x\right)\right]. \tag{1.113}$$

Finally, then,

$$-\frac{ie^3}{2\hbar}\int d\omega' d\omega''(\bar{\psi}(x')K_\mu(x'-x,x-x'')\psi(x''))_1=\frac{\alpha}{4\pi}\log\frac{\kappa}{2k_{\min}}\frac{1}{\kappa^2}\Box^2\int[F_0(x-x')+F_1(x-x')]j_\mu(x')d\omega'$$
$$+\frac{\alpha}{4\pi}\frac{1}{\kappa^2}\Box^2\int[\tfrac{1}{2}F_0(x-x')+F_1(x-x')+\tfrac{1}{2}G(x-x')]j_\mu(x')d\omega'+\frac{\alpha}{2\pi}c\frac{\partial}{\partial x_\nu}\int F_0(x-x')m_{\mu\nu}(x')d\omega', \tag{1.114}$$

in which

$$m_{\mu\nu}(x)=\frac{e}{2\kappa}(\bar{\psi}(x)\sigma_{\mu\nu}\psi(x))_1$$
$$=\frac{e}{2\kappa}\tfrac{1}{2}[\bar{\psi}(x)\sigma_{\mu\nu}\psi(x)-\bar{\psi}'(x)\sigma_{\mu\nu}\psi'(x)]. \quad (1.115)$$

Expressed in the same notation, the first term of (1.58) is (see II (2.44)),

$$\frac{i}{2\hbar c^2}\int d\omega'\epsilon(x-x')[j_\mu(x),j_\nu(x')]_0\delta A_\nu(x')=-\frac{\alpha}{4\pi}\frac{1}{\kappa^2}\Box^2\int[F_1(x-x')-\tfrac{1}{3}F_2(x-x')]j_\mu(x')d\omega', \quad (1.116)$$

from which we have omitted the charge renormalization term, with the understanding that the value of e is to be correspondingly altered. A rederivation of this result, employing methods akin to those presented in this paper, is given in the Appendix. Evidently the new contributions to the one particle current operator, as given in (1.114), are of the same general nature as the previously considered effect, (1.116), with the exception of the last term in (1.114). This is an addition to the current vector of the form

$$c(\partial/\partial x)_\nu\delta m_{\mu\nu}(x), \quad (1.117)$$

where

$$\delta m_{\mu\nu}(x)=\alpha/2\pi\int F_0(x-x')m_{\mu\nu}(x')d\omega'. \quad (1.118)$$

A current vector of this type can be interpreted as a dipole current, derived from an antisymmetrical dipole tensor $\delta m_{\mu\nu}$ which combines electric and magnetic dipole moment densities. The tensor $m_{\mu\nu}$ is that characteristic of the Dirac theory, in which intrinsic dipole moments are related to the antisymmetrical spin tensor $\sigma_{\mu\nu}$, the factor of proportionality being

$$\mu_0=e/2\kappa=e\hbar/2mc, \quad (1.119)$$

the Bohr magneton. According to (1.118), the correction to the dipole tensor at a point involves an average of $m_{\mu\nu}$ over the vicinity of that point. If all quantities are slowly varying, relative to $\hbar/mc$ and $\hbar/mc^2$ as units of length and time, an expansion in ascending powers of $\Box^2$ can be constructed, as in II (2.47). For this purpose, it is sufficient to expand the denominator in the first form of (1.112), thus obtaining

$$F_n(x)=\frac{1}{2n+1}\delta(x)+\frac{1}{(2n+1)(2n+3)}\frac{1}{2\kappa^2}\Box^2\delta(x)+\cdots. \quad (1.120)$$

Hence,

$$\delta m_{\mu\nu}(x)=\frac{\alpha}{2\pi}\left[m_{\mu\nu}(x)+\frac{1}{6\kappa^2}\Box^2 m_{\mu\nu}(x)+\cdots\right], \quad (1.121)$$

and, under conditions that permit the neglect of all but the first term in this series, an electron will act as though it possessed an additional spin magnetic moment[3]

$$\delta\mu=(\alpha/2\pi)\mu_0. \quad (1.122)$$

The comparison of this prediction with experiment will be discussed in the sequel to this paper.

The final result for the second-order correction to the one particle current operator is

$$(\delta j_\mu^{(2)}(x))_{1,0}=\frac{\alpha}{4\pi}\log\frac{\kappa}{2k_{\min}}\frac{1}{\kappa^2}\Box^2\int[F_0(x-x')+F_1(x-x')]j_\mu(x')d\omega'$$
$$+\frac{\alpha}{4\pi}\frac{1}{\kappa^2}\Box^2\int[\tfrac{1}{2}F_0(x-x')+\tfrac{1}{3}F_2(x-x')+\tfrac{1}{2}G(x-x')]j_\mu(x')d\omega'+c\frac{\partial}{\partial x_\nu}\delta m_{\mu\nu}(x). \quad (1.123)$$

Under conditions of slow variation $((1/\kappa^2)\Box^2 j_\mu,\ m_{\mu\nu}\ll j_\mu, m_{\mu\nu})$, this reduces to

$$(\delta j_\mu^{(2)}(x))_{1,0}=\frac{\alpha}{3\pi}\left(\log\frac{\kappa}{2k_{\min}}+\frac{17}{40}\right)\frac{1}{\kappa^2}\Box^2 j_\mu(x)+\frac{\alpha}{2\pi}c\frac{\partial}{\partial x_\nu}m_{\mu\nu}(x), \quad (1.124)$$

in virtue of (1.120), and the analogous expansion of $G(x)$,

$$G(x)=-\frac{1}{5\kappa^2}\Box^2\delta(x)+\cdots. \quad (1.125)$$

It will be noted that the total charge computed from $(\delta j_\mu^{(2)}(x))_{1,0}$ is zero, in agreement with evident charge conservation requirements, and the formal property that the operator of total charge commutes with all

[3] This result was announced at the January, 1948 meeting of the American Physical Society. The formula is misprinted in a published note, J. Schwinger, Phys. Rev. 73, 416 (1948). The misprint has unfortunately been copied by L. Rosenfeld in his book, *Nuclear Forces* (Interscience Publishers, Inc., New York, 1949), p. 438.

one-particle operators. The apparent contradiction between these statements and the existence of the charge renormalization term is discussed in the Appendix, where it is shown that a compensating charge is created at infinity.

Our result, (1.123), is marred only by the appearance of the logarithmic divergence associated with zero frequency quanta. It should be remarked, however, that $(\delta j_\mu^{(2)}(x))_{1,0}$ is not a complete description of the radiative corrections under discussion. In order to measure the correction to the current, it is necessary to impose an external field. This will induce the emission of quanta, as described by $\delta j_\mu^{(1)}(x)$, among the effects of which is a compensating low frequency divergence. It will be apparent that, as a consequence of the "infra-red catastrophe," the second-order corrections to particle electromagnetic properties cannot be completely stated without regard for the manner of exhibiting them by an external field. We therefore turn to a discussion of the behavior of a single particle in an external field, as modified by the vacuum fluctuations of the electromagnetic field.

2. RADIATIVE CORRECTIONS TO ELECTRON SCATTERING

We shall now be concerned with the interaction of three systems—the matter field, the electromagnetic field, and a given current distribution. The latter may be associated with a nucleus or a macroscopic apparatus, two situations in which the reaction on the current distribution may have a negligible effect. A description of this state of affairs, in the interaction representation, is given by

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left[-\frac{1}{c}(j_\mu(x)+J_\mu(x))A_\mu(x)\right]\Psi[\sigma], \tag{2.1a}$$

$$\left[\frac{\partial A_\mu(x')}{\partial x_\mu'}-\frac{1}{c}\int_\sigma D(x'-x)(j_\mu(x)+J_\mu(x))d\sigma_\mu\right]\Psi[\sigma]=0, \tag{2.1b}$$

where $j_\mu(x)$ and $J_\mu(x)$ are the current vectors associated with the matter field and the external system, respectively. Both current distributions are coupled to the electromagnetic field, as characterized by $A_\mu(x)$. An equally valid way of stating matters is in terms of an external electromagnetic field acting on the matter field current distribution:

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left[-\frac{1}{c}j_\mu(x)(A_\mu(x)+A_\mu^{(e)}(x))\right]\Psi[\sigma], \tag{2.2a}$$

$$\left[\frac{\partial A_\mu(x')}{\partial x_\mu'}-\frac{1}{c}\int_\sigma D(x'-x)j_\mu(x)d\sigma_\mu\right]\Psi[\sigma]=0, \tag{2.2b}$$

where

$$\square^2 A_\mu^{(e)}(x)=-\frac{1}{c}J_\mu(x),\quad \frac{\partial A_\mu^{(e)}(x)}{\partial x_\mu}=0. \tag{2.3}$$

The equivalence of the two descriptions is established by showing that (2.2) is obtained from (2.1) by a canonical transformation, namely,

$$\Psi[\sigma]\rightarrow e^{-iJ[\sigma]}\Psi[\sigma], \tag{2.4}$$

with $J[\sigma]$ determined by

$$\hbar c\frac{\delta J[\sigma]}{\delta\sigma(x)}=-\frac{1}{c}J_\mu(x)A_\mu(x). \tag{2.5}$$

The functional $J[\sigma]$ is explicitly exhibited as

$$J[\sigma]=-\frac{1}{\hbar c^2}\int_{-\infty}^{\sigma}J_\mu(x')A_\mu(x')d\omega', \tag{2.6}$$

in which the choice of lower limit corresponds to selecting the retarded potentials for the electromagnetic field generated by the given current distribution. The equation of motion satisfied by the new state vector is

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}+i\hbar c e^{iJ[\sigma]}\frac{\delta e^{-iJ[\sigma]}}{\delta\sigma(x)}\Psi[\sigma]=\left[-\frac{1}{c}(j_\mu(x)+J_\mu(x))e^{iJ[\sigma]}A_\mu(x)e^{-iJ[\sigma]}\right]\Psi[\sigma]. \tag{2.7}$$

Now

$$e^{iJ[\sigma]}A_\mu(x)e^{-iJ[\sigma]}=A_\mu(x)+i[J[\sigma],A_\mu(x)]-\tfrac{1}{2}[J[\sigma],[J[\sigma],A_\mu(x)]]+\cdots$$
$$=A_\mu(x)-\frac{1}{c}\int_{-\infty}^{\sigma}D(x-x')J_\mu(x')d\omega'$$
$$=A_\mu(x)+A_\mu^{(e)}(x), \tag{2.8}$$

in which the series ends after two terms since the components of $J_\mu(x)$ are mutually commutative, in view of the prescribed nature of this current distribution. It is easily seen that

$$A_\mu^{(e)}(x)=-\frac{1}{c}\int_{-\infty}^{\sigma}D(x-x')J_\mu(x')d\omega' \tag{2.9}$$

obeys (2.3). Indeed,

$$\Box^2 A_\mu^{(e)}(x)=-\frac{1}{c}\frac{\partial}{\partial x_\nu}\int_{-\infty}^{\sigma}\frac{\partial D(x-x')}{\partial x_\nu'}J_\mu(x')d\omega'$$
$$=-\frac{1}{c}\int_\sigma d\sigma_\nu'\frac{\partial D(x-x')}{\partial x_\nu'}J_\mu(x')$$
$$=-\frac{1}{c}J_\mu(x), \tag{2.10}$$

and

$$\frac{\partial A_\mu^{(e)}(x)}{\partial x_\mu}=-\frac{1}{c}\int_{-\infty}^{\sigma}d\omega'\frac{\partial}{\partial x_\mu'}(D(x-x')J_\mu(x'))$$
$$=0. \tag{2.11}$$

Furthermore,

$$i\hbar c e^{iJ[\sigma]}\frac{\delta e^{-iJ[\sigma]}}{\delta\sigma(x)}=\hbar c\frac{\delta J[\sigma]}{\delta\sigma(x)}+\frac{i\hbar c}{2}\left[J[\sigma],\frac{\delta J[\sigma]}{\delta\sigma(x)}\right]+\cdots$$
$$=-\frac{1}{c}J_\mu(x)A_\mu(x)-\frac{1}{2c}J_\mu(x)A_\mu^{(e)}(x), \tag{2.12}$$

and the transformed equation of motion therefore reads

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\left[-\frac{1}{c}j_\mu(x)(A_\mu(x)+A_\mu^{(e)}(x))-\frac{1}{2c}J_\mu(x)A_\mu^{(e)}(x)\right]\Psi[\sigma], \tag{2.13}$$

which is equivalent to (2.2a), since the term $-(1/2c)J_\mu(x)A_\mu^{(e)}(x)$, describing the self-action of the given current distribution, has no dynamical consequences and can be omitted.

In a similar way, the supplementary condition (2.1b) is transformed into

$$\left[e^{iJ[\sigma]}\frac{\partial A_\mu(x')}{\partial x_\mu'}e^{-iJ[\sigma]}-\frac{1}{c}\int_\sigma D(x'-x)(j_\mu(x)+J_\mu(x))d\sigma_\mu\right]\Psi[\sigma]=0, \tag{2.14}$$

wherein

$$e^{iJ[\sigma]}\frac{\partial A_\mu(x')}{\partial x_\mu'}e^{-iJ[\sigma]}=\frac{\partial A_\mu(x')}{\partial x_\mu'}+i\left[J[\sigma],\frac{\partial A_\mu(x')}{\partial x_\mu'}\right]=\frac{\partial A_\mu(x')}{\partial x_\mu'}-\frac{1}{c}\int_{-\infty}^{\sigma}\frac{\partial D(x'-x'')}{\partial x_\mu'}J_\mu(x'')d\omega''. \tag{2.15}$$

However,

$$-\frac{1}{c}\int_{-\infty}^{\sigma}\frac{\partial D(x'-x'')}{\partial x_\mu'}J_\mu(x'')d\omega''=\frac{1}{c}\int_{-\infty}^{\sigma}\frac{\partial}{\partial x_\mu''}(D(x'-x'')J_\mu(x''))d\omega''=\frac{1}{c}\int_\sigma D(x'-x)J_\mu(x)d\sigma_\mu, \tag{2.16}$$

which verifies (2.2b).

One can bring (2.2) into a form which enables the results of the previous section to be utilized. The mass renormalization transformation

$$\Psi[\sigma]\rightarrow W[\sigma]\Psi[\sigma],\quad i\hbar c\frac{[\delta W\sigma]}{\delta\sigma(x)}=\mathcal{H}_{1,0}(x)W[\sigma], \tag{2.17}$$

replaces (2.2a) with

$$ihc\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=[\mathcal{K}(x)+\mathcal{H}^{(e)}(x)]\Psi[\sigma], \tag{2.18}$$

where (see (1.3))

$$\mathcal{K}(x)=\mathcal{H}(x)-\mathcal{H}_{1,0}(x), \tag{2.19}$$

$$\mathcal{H}^{(e)}(x)=-\frac{1}{c}j_\mu(x)A_\mu^{(e)}(x), \tag{2.20}$$

and the Dirac equation for $\psi(x)$ now involves the experimental mass. The further transformation

$$\Psi[\sigma]=U[\sigma]\Phi[\sigma], \tag{2.21}$$

where

$$ihc\frac{\delta U[\sigma]}{\delta\sigma(x)}=\mathcal{K}(x)U[\sigma],\quad U[-\infty]=1, \tag{2.22}$$

is the analog of (1.8), save that $\Phi[\sigma]$ varies in the presence of an external field,

$$\begin{aligned}ihc\frac{\delta\Phi[\sigma]}{\delta\sigma(x)}&=U^{-1}[\sigma]\mathcal{H}^{(e)}(x)U[\sigma]\Phi[\sigma]\\&=-\frac{1}{c}\mathbf{j}_\mu(x)A_\mu^{(e)}(x)\Phi[\sigma],\end{aligned} \tag{2.23}$$

in response to the coupling with the current operator $\mathbf{j}_\mu(x)$. The latter contains the modifications produced by the vacuum electromagnetic field. The supplementary condition (2.2b) appears as

$$\left[U^{-1}[\sigma]\frac{\partial A_\mu(x')}{\partial x_\mu'}U[\sigma]-\frac{1}{c}\int_\sigma D(x'-x)\mathbf{j}_\mu(x)d\sigma_\mu\right]\Phi[\sigma]=0, \tag{2.24}$$

in consequence of these transformations. However,

$$\begin{aligned}U^{-1}[\sigma]\frac{\partial A_\mu(x')}{\partial x_\mu'}U[\sigma]-\frac{\partial A_\mu(x')}{\partial x_\mu'}&=\int_{-\infty}^{\sigma}d\omega''\frac{\delta}{\delta\sigma''(x'')}\left(U^{-1}[\sigma'']\frac{\partial A_\mu(x')}{\partial x_\mu'}U[\sigma'']\right)\\&=\frac{i}{\hbar c^2}\int_{-\infty}^{\sigma}d\omega''U^{-1}[\sigma'']\left[\frac{\partial A_\mu(x')}{\partial x_\mu'},A_\nu(x'')\right]j_\nu(x'')U[\sigma'']=\frac{1}{c}\int_{-\infty}^{\sigma}d\omega''\frac{\partial}{\partial x_\mu''}(D(x'-x'')\mathbf{j}_\mu(x''))\\&=\frac{1}{c}\int_\sigma D(x'-x)\mathbf{j}_\mu(x)d\sigma_\mu,\end{aligned} \tag{2.25}$$

so that the supplementary condition associated with (2.23) is simply

$$\frac{\partial A_\mu(x')}{\partial x_\mu'}\Phi[\sigma]=0. \tag{2.26}$$

As the first application of (2.23), we shall consider the scattering of an electron produced by its interaction with an external field, in which the latter is regarded as a small perturbation.[4] We shall restrict the external potential to be that of a time independent field, which will eventually be specialized to the Coulomb field of a stationary nucleus.

A solution of (2.23) can be constructed in the form

$$\Phi[\sigma]=R[\sigma]\Phi_1, \tag{2.27}$$

where

$$ihc\frac{\delta R[\sigma]}{\delta\sigma(x)}=H(x)R[\sigma], \tag{2.28}$$

and

$$R[\sigma]\to 1,\quad \sigma\to-\infty. \tag{2.29}$$

[4] Radiative corrections to scattering have been discussed by many authors. That a finite correction is obtained after a renormalization of charge and mass was independently observed by Z. Koba and S. Tomonaga, Prog. Theor. Phys. 3, 290 (1948); H. W. Lewis, Phys. Rev. 73, 173 (1948); and J. Schwinger, Phys. Rev. 73, 416 (1948). See also R. P. Feynman, Phys. Rev. 74, 1430 (1948).

The state vector Φ_1 characterizes the initial state of the system, composed of one electron with definite energy and momentum, and no light quanta. The total probability, per unit time, that a scattering process occurs, can be obtained by evaluating the time rate of decrease of the probability that the system remain in the initial state,

$$w=-c\int dv\frac{\delta}{\delta\sigma(x)}|(\Phi_1,\Phi[\sigma])|^2=-c\int dv\frac{\delta}{\delta\sigma(x)}|(\Phi_1,R[\sigma]\Phi_1)|^2. \tag{2.30}$$

The integration is extended over the surface $t=$const., with dv the three-dimensional volume element. Now

$$i\hbar c\frac{\delta}{\delta\sigma(x)}|(\Phi_1,R[\sigma]\Phi_1)|^2=(\Phi_1,R^{-1}[\sigma]\Phi_1)(\Phi_1,H(x)R[\sigma]\Phi_1)-(\Phi_1,R[\sigma]\Phi_1)(\Phi_1,R^{-1}[\sigma]H(x)\Phi_1). \tag{2.31}$$

In view of the treatment of $H(x)$ as a small perturbation, it is sufficient to write

$$R[\sigma]=1-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}H(x')d\omega',\quad R^{-1}[\sigma]=1+\frac{i}{\hbar c}\int_{-\infty}^{\sigma}H(x')d\omega'. \tag{2.32}$$

It will also be useful to introduce

$$H'(x)=H(x)-(\Phi_1,H(x)\Phi_1), \tag{2.33}$$

which possesses a vanishing diagonal matrix element for the initial state, and obtain

$$R[\sigma]=\exp\left[-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}(1|H(x')|1)d\omega'\right]\left(1-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}H'(x')d\omega'\right). \tag{2.34}$$

The phase factor evidently has no effect in (2.31), and can be omitted. The latter is also unaffected if $H(x)$ is replaced by $H'(x)$. Hence to the accuracy of first-order perturbation theory, we have

$$-\frac{\delta}{\delta\sigma(x)}|(\Phi_1,R[\sigma]\Phi_1)|^2=\frac{1}{\hbar^2c^2}\left(1\left|H'(x)\int_{-\infty}^{\sigma}H'(x')d\omega'+\int_{-\infty}^{\sigma}H'(x')d\omega'H'(x)\right|1\right), \tag{2.35}$$

and

$$w=\frac{1}{\hbar^2c}\int dvdv'\left(1\left|H'(x)\int_{\infty}^{x_0}H'(x')dx_0'+\int_{-\infty}^{x_0}H'(x')dx_0'H'(x)\right|1\right). \tag{2.36}$$

We may now remark that a diagonal matrix element for a state of definite energy must be invariant with respect to time displacements, whence

$$\int dvdv'\left(1\left|\int_{\infty}^{x_0}H'(x')dx_0'H'(x)\right|1\right)=\int dvdv'\left(1\left|H'(x)\int_{x_0}^{\infty}H'(x')dx_0'\right|1\right), \tag{2.37}$$

and

$$w=\frac{1}{\hbar^2c}\int dvdv'\left(1\left|H'(x)\int_{-\infty}^{\infty}H'(x')dx_0'\right|1\right). \tag{2.38}$$

This result is perfectly equivalent to the more conventional perturbation formula in which the rate of transition from the initial state is expressed as a sum of transition rates to all possible final states of equal energy. The energy conservation law is here expressed by the time integration, and the summation over all states other than the original is provided for by the removal from $H(x)$ of the diagonal matrix element. Our basic formula for calculating the transition rate for scattering of a particle by a time independent potential is thus

$$w=\frac{1}{\hbar^2c^3}\int dvdv'A_\mu^{(e)}(\mathbf{r})A_\nu^{(e)}(\mathbf{r}')\left(1\left|\mathbf{j}_\mu(x)\int_{-\infty}^{\infty}\mathbf{j}_\nu(x')dx_0'\right|1\right). \tag{2.39}$$

We have not indicated that the diagonal matrix element is to be subtracted from $\mathbf{j}_\mu(x)$, since it is sufficient to remove, in the final result, those transitions in which no change of state occurs.

We have shown in the first section that, to the second order in e,

$$\mathbf{j}_\mu(x)=j_\mu(x)+\delta j_\mu^{(1)}(x)+\delta j_\mu^{(2)}(x), \tag{2.40}$$

where

$$\delta j_\mu^{(1)}(x)=\frac{i}{2\hbar c^2}\int_{-\infty}^{\infty}\epsilon(x-x')[j_\mu(x),j_\nu(x')]_1A_\nu(x')d\omega'$$

$$=\frac{ie^2}{\hbar}\int_{-\infty}^{\infty}[\bar\psi(x)\gamma_\mu\bar S(x-x')\gamma_\nu\psi(x')+\bar\psi(x')\gamma_\nu\bar S(x'-x)\gamma_\mu\psi(x)]_1A_\nu(x')d\omega', \tag{2.41}$$

and

$$(\delta j_\mu^{(2)}(x))_{1,0}=iec\int_{-\infty}^{\infty}[\bar\psi(x')\Gamma_\mu(x-x')\psi(x')]_1d\omega'. \tag{2.42}$$

Here

$$\Gamma_\mu(x)=\frac{\alpha}{4\pi}\gamma_\mu\log\frac{\kappa}{2k_{\min}}\frac{1}{\kappa^2}\Box^2(F_0(x)+F_1(x))+\frac{\alpha}{4\pi}\gamma_\mu\frac{1}{\kappa^2}\Box^2(\tfrac{1}{2}F_0(x)+\tfrac{1}{3}F_2(x)+\tfrac{1}{2}G(x))-i\frac{\alpha}{4\pi}\sigma_{\mu\nu}\frac{1}{\kappa}\frac{\partial}{\partial x_\nu}F_0(x). \tag{2.43}$$

It is only the indicated portion of $\delta j_\mu^{(2)}(x)$, referring to one particle and no light quanta, that need be retained to compute the second-order correction to the scattering cross section for an external field, since only this part of $\delta j_\mu^{(2)}(x)$ is coherent with $j_\mu(x)$.

The total rate of transition from the initial state can now be written as

$$w=w_0+w_1, \tag{2.44}$$

where

$$w_0=\frac{1}{\hbar^2c^3}\int dvdv'A_\mu^{(e)}(\mathbf{r})A_\nu^{(e)}(\mathbf{r}')\left(1\left|(j_\mu(x)+(\delta j_\mu^{(2)}(x))_{1,0})\int_{-\infty}^{\infty}(j_\nu(x')+(\delta j_\nu^{(2)}(x'))_{1,0})dx_0'\right|1\right) \tag{2.45}$$

describes the rate of radiationless scattering, while

$$w_1=\frac{1}{\hbar^2c^3}\int dvdv'A_\mu^{(e)}(\mathbf{r})A_\nu^{(e)}(\mathbf{r}')\left(1\left|\delta j_\mu^{(1)}(x)\int_{-\infty}^{\infty}\delta j_\nu^{(1)}(x')dx_0'\right|1\right) \tag{2.46}$$

accounts for scattering that is accompanied by single quantum emission.

To indicate the manner in which the perturbation formulas are to be used, we consider the evaluation of

$$\int_{-\infty}^{\infty}(1|j_\mu(x)j_\nu(x')|1)dx_0'. \tag{2.47}$$

This can be written as

$$-e^2c^2\int_{-\infty}^{\infty}(1|\bar\psi(x)\gamma_\mu\psi(x)\bar\psi(x')\gamma_\nu\psi(x')|1)dx_0', \tag{2.48}$$

in which it is understood that one omits the processes in which $\bar\psi(x)\psi(x)$, or $\bar\psi(x')\psi(x')$ induces no change in state. Now $\psi(x')$ can either annul the original electron, in which case $\bar\psi(x')\psi(x')$ causes an electron transition to some final state, or $\psi(x')$ generates a positron, in which event $\bar\psi(x')\psi(x')$ induces the creation of a pair. However, the latter process is incompatible with the energy conservation that is enforced by the time integration, and can therefore be omitted. Hence, it only occurs that $\psi(x')$ annihilates the original electron, whence $\psi(x')\Phi_1$ is a multiple of the vacuum state vector. The same comment applies to $\bar\psi^\dagger(x)\Phi_1=\gamma_4\psi(x)\Phi_1$. Therefore, only the vacuum expectation value of the operator $\psi(x)\bar\psi(x')$ is required in (2.48). Furthermore, since only one state of the matter field is initially excited, as described by the wave function ue^{ipx}, we arrive at the result

$$\int_{-\infty}^{\infty}(1|j_\mu(x)j_\nu(x')|1)dx_0'=\frac{e^2c^2}{(2\pi)^2}\int(dq)\delta(q_0-p_0)\delta(q^2+\kappa^2)\bar u\gamma_\mu(i\gamma q-\kappa)\gamma_\nu ue^{i(\mathbf{p}-\mathbf{q})\cdot(\mathbf{r}'-\mathbf{r})}, \tag{2.49}$$

on employing the relation

$$\langle\psi_\alpha(x)\bar\psi_\beta(x')\rangle_0=-iS_{\alpha\beta}^{(+)}(x-x')=-\frac{1}{(2\pi)^3}\int_{q_0>0}(dq)\delta(q^2+\kappa^2)(i\gamma q-\kappa)_{\alpha\beta}e^{iq(x-x')}. \tag{2.50}$$

Before further simplifying this expression, we shall consider the analogous evaluations of

$$\int_{-\infty}^{\infty}(1|j_\mu(x)(\delta j_\nu^{(2)}(x'))_{1,0}|1)dx_0' \tag{2.51a}$$

and

$$\int_{-\infty}^{\infty}(1|(\delta j_\mu^{(2)}(x))_{1,0}j_\nu(x')|1)dx_0' \tag{2.51b}$$

which describe the radiationless corrections to the scattering process. Now (2.51a) can be written

$$-e^2c^2\int_{-\infty}^{\infty}dx_0'\int d\omega''(1|\bar{\psi}(x)\gamma_\mu\psi(x)\bar{\psi}(x'')\Gamma_\nu(x'-x'')\psi(x'')|1), \tag{2.52}$$

which, according to the arguments presented in connection with (2.47), becomes

$$-e^2c^2\int_{-\infty}^{\infty}dx_0'\int d\omega''\bar{u}\gamma_\mu\langle\psi(x)\bar{\psi}(x'')\rangle_0\Gamma_\nu(x'-x'')ue^{ip(x''-x)}$$
$$=\frac{e^2c^2}{(2\pi)^3}\int_{-\infty}^{\infty}dx_0'\int d\omega''\int_{q_0>0}(dq)\delta(q^2+\kappa^2)\bar{u}\gamma_\mu(i\gamma q-\kappa)\Gamma_\nu(x'-x'')ue^{i(p-q)(x''-x)}. \tag{2.53}$$

On introducing the Fourier transform of $\Gamma_\nu(x)$:

$$\Gamma_\nu(p-q)=\int e^{-i(p-q)x}\Gamma_\nu(x)d\omega, \tag{2.54}$$

we obtain

$$\int_{-\infty}^{\infty}(1|j_\mu(x)(\delta j_\nu^{(2)}(x'))_{1,0}|1)dx_0'=\frac{e^2c^2}{(2\pi)^2}\int(dq)\delta(q_0-p_0)\delta(q^2+\kappa^2)\bar{u}\gamma_\mu(i\gamma q-\kappa)\Gamma_\nu(p-q)ue^{i(\mathbf{p}-\mathbf{q})\cdot(\mathbf{r}'-\mathbf{r})}. \tag{2.55}$$

The result of combining (2.49) and (2.55) with the analogous evaluation of (2.51b) is expressed by

$$w_0=\frac{1}{(2\pi)^2}\frac{e^2}{\hbar^2c}\int dq_0(d\mathbf{q})\delta(q_0-p_0)\delta(q^2+\kappa^2)\int e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}}A_\mu^{(e)}(\mathbf{r})dv\int e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}'}A_\nu^{(e)}(\mathbf{r}')dv'$$
$$\times\bar{u}(\gamma_\mu+\Gamma_\mu(q-p))(i\gamma q-\kappa)(\gamma_\nu+\Gamma_\nu(p-q))u. \tag{2.56}$$

On performing the integration with respect to q_0 and $|\mathbf{q}|$, we obtain w_0 in the form of an integral extended over all directions of the vector $\mathbf{q}$, other than the incident direction:

$$w_0=\frac{1}{8\pi^2}\frac{e^2}{\hbar^2c}\int d\Omega|\mathbf{p}|\int e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}}A_\mu^{(e)}(\mathbf{r})dv\int e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}'}A_\nu^{(e)}(\mathbf{r}')dv'\bar{u}(\gamma_\mu+\Gamma_\mu(q-p))(i\gamma q-\kappa)(\gamma_\nu+\Gamma_\nu(p-q))u. \tag{2.57}$$

This must be interpreted as the rate of transition from the initial state, expressed as the probability per unit time for a deflection into an arbitrary element of solid angle. A further simplification can be introduced by averaging (2.57) with respect to the two spin states in which the incident electron may occur. For this purpose, we require the average of $u_\alpha\bar{u}_\beta$ for the two polarization states associated with a given energy and momentum. It can be inferred from the anticommutator

$$\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}=\frac{1}{i}S_{\alpha\beta}(x-x')=-\frac{1}{(2\pi)^3}\int(dp)\delta(p^2+\kappa)\epsilon(p)(i\gamma p-\kappa)_{\alpha\beta}e^{ip(x-x')}, \tag{2.58}$$

which exhibits, with equal weight, the contributions of all states of a particle, that

$$\langle u_\alpha\bar{u}_\beta\rangle=A(i\gamma p-\kappa)_{\alpha\beta}, \tag{2.59}$$

for a state with wave number four-vector p_μ. The constant A is conveniently evaluated for our purpose in terms of the expectation value of the particle flux vector in the initial state,

$$\mathbf{S}^{(inc)}=(1|ic(\bar{\psi}(x)\boldsymbol{\gamma}\psi(x))_1|1)$$
$$=ic\bar{u}\boldsymbol{\gamma}u. \tag{2.60}$$

Thus,

$$\mathbf{S}^{(inc)}=ic\boldsymbol{\gamma}_{\beta\alpha}u_\alpha\bar{u}_\beta\rightarrow icA\,Tr\boldsymbol{\gamma}(i\gamma p-\kappa)$$
$$=-4cA\mathbf{p}, \tag{2.61}$$

so that

$$\langle u_\alpha \bar{u}_\beta\rangle = -\frac{1}{4c}\frac{|\mathbf{S}^{(inc)}|}{|\mathbf{p}|}(i\gamma p-\kappa)_{\alpha\beta}. \tag{2.62}$$

This leads to the following expression for the total rate of transition from the initial state,

$$w_0=\frac{1}{8\pi^2}\frac{e^2}{\hbar^2c^2}|\mathbf{S}^{(inc)}|\int d\Omega\left|\int e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}}\frac{Ze}{4\pi r}dv\right|^2 \tfrac{1}{4}Tr[(i\gamma p-\kappa)(\gamma_4+\Gamma_4(q-p))(i\gamma q-\kappa)(\gamma_4+\Gamma_4(p-q))], \tag{2.63}$$

in which we have also specialized to the Coulomb potential of a stationary nucleus. We may now infer that the differential cross section for radiationless scattering through the angle ϑ into a unit solid angle is

$$\frac{d\sigma_0(\vartheta)}{d\Omega}=2\left[\frac{Z\alpha}{(\mathbf{p}-\mathbf{q})^2}\right]^2\tfrac{1}{4}Tr[(i\gamma p-\kappa)(\gamma_4+\Gamma_4(q-p))(i\gamma q-\kappa)(\gamma_4+\Gamma_4(p-q))]. \tag{2.64}$$

The Fourier transform of Γ_4 is conveniently written in the form

$$\Gamma_4(p-q)=-\frac{\alpha}{4\pi}\gamma_4\left[4\lambda^2A(\lambda)+\frac{i}{\kappa}\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{q})F_0(\lambda)\right], \tag{2.65}$$

where

$$A(\lambda)=\log\frac{\kappa}{2k_{\min}}(F_0(\lambda)+F_1(\lambda))+\tfrac{1}{2}(F_0(\lambda)+\tfrac{2}{3}F_2(\lambda)+G(\lambda)). \tag{2.66}$$

Here

$$\lambda=\frac{|\mathbf{p}-\mathbf{q}|}{2\kappa}=\frac{|\mathbf{p}|}{\kappa}\sin\frac{\vartheta}{2}, \tag{2.67}$$

and

$$F_0(\lambda)=\int_0^1\frac{dv}{1+\lambda^2(1-v^2)}=\frac{\log((1+\lambda^2)^{\frac{1}{2}}+\lambda)}{(1+\lambda^2)^{\frac{1}{2}}\lambda} \tag{2.68a}$$

$$F_1(\lambda)=\int_0^1\frac{v^2dv}{1+\lambda^2(1-v^2)}=\left(1+\frac{1}{\lambda^2}\right)F_0(\lambda)-\frac{1}{\lambda^2} \tag{2.68b}$$

$$F_2(\lambda)=\int_0^1\frac{v^4dv}{1+\lambda^2(1-v^2)}=\left(1+\frac{1}{\lambda^2}\right)F_1(\lambda)-\frac{1}{3\lambda^2}. \tag{2.68c}$$

The more complicated transform, $G(\lambda)$ is not required in the following development. The trace of the Dirac matrices contained in (2.64) is easily computed:

$$\tfrac{1}{4}Tr[(i\gamma p-\kappa)(\gamma_4+\Gamma_4(q-p))(i\gamma q-\kappa)(\gamma_4+\Gamma_4(p-q))]=2(p_0^2-\kappa^2\lambda^2)\left(1-\frac{2\alpha}{\pi}\lambda^2A(\lambda)\right)-\frac{2\alpha}{\pi}\kappa^2\lambda^2F_0(\lambda), \tag{2.69}$$

whence

$$\frac{d\sigma_0(\vartheta)}{d\Omega}=\left(\frac{Z\alpha}{2|\mathbf{p}|\beta}\csc^2\frac{\vartheta}{2}\right)^2\left(1-\beta^2\sin^2\frac{\vartheta}{2}\right)\left[1-\frac{2\alpha}{\pi}\lambda^2A(\lambda)-\frac{\alpha}{\pi}\frac{\kappa^2}{p_0^2-\kappa^2\lambda^2}\lambda^2F_0(\lambda)\right] \tag{2.70}$$

in which $\beta=|\mathbf{p}|/p_0$ is the speed of the particle relative to c.

To evaluate the rate at which transitions occur accompanied by radiation, we consider

$$\left(1\left|\delta j_\mu^{(1)}(x)\int_{-\infty}^{\infty}\delta j_\nu^{(1)}(x')dx_0'\right|1\right)=-\frac{e^4}{\hbar^2}\int d\omega''d\omega'''\left(1\left|[\bar\psi(x)\gamma_\mu\bar S(x-x'')\gamma_\lambda\psi(x'')+\bar\psi(x'')\gamma_\lambda\bar S(x''-x)\gamma_\mu\psi(x)]_1\right.\right.$$
$$\left.\left.\times\int_{-\infty}^{\infty}dx_0'[\bar\psi(x')\gamma_\nu\bar S(x'-x''')\gamma_\sigma\psi(x''')+\bar\psi(x''')\gamma_\sigma\bar S(x'''-x')\gamma_\nu\psi(x')]_1A_\lambda(x'')A_\sigma(x''')\right|1\right). \tag{2.71}$$

Since the state vector Φ_1 is characterized by an absence of quanta, only the following vacuum expectation value is required for the electromagnetic field,

$$\langle A_\lambda(x'')A_\sigma(x''')\rangle_0=i\hbar c\delta_{\lambda\sigma}D^{(+)}(x''-x''')=\frac{\hbar c}{(2\pi)^3}\delta_{\lambda\sigma}\int_{k_0>0}(dk)\delta(k^2)e^{ik(x''-x''')}. \tag{2.72}$$

The matter field operators are treated as before, with the result

$$\left(1\left|\delta j_\mu^{(1)}(x)\int_{-\infty}^{\infty}\delta j_\mu^{(1)}(x')dx_0'\right|1\right)=\frac{e^4}{\hbar^2}\frac{\hbar c}{(2\pi)^6}\int_{q_0,k_0>0}(dq)\delta(q^2+\kappa^2)(dk)\delta(k^2)\int_{-\infty}^{\infty}dx_0'e^{i(p-q-k)(x'-x)}$$
$$\times\bar{u}(\gamma_\mu\bar{S}(q+k)\gamma_\lambda+\gamma_\lambda\bar{S}(p-k)\gamma_\mu)(i\gamma q-\kappa)(\gamma_\nu\bar{S}(p-k)\gamma_\lambda+\gamma_\lambda\bar{S}(q+k)\gamma_\nu)u. \quad (2.73)$$

Here

$$\bar{S}(q+k)=\int e^{-i(q+k)x}\bar{S}(x)d\omega=\frac{i\gamma(q+k)-\kappa}{2qk} \quad (2.74)$$

and

$$\bar{S}(p-k)=-\frac{i\gamma(p-k)-\kappa}{2pk} \quad (2.75)$$

are Fourier transforms of $\bar{S}(x)$. The integration with respect to x_0' imposes the energy conservation law

$$p_0=q_0+k_0, \quad (2.76)$$

which is evidently that of a light quantum emission process. The integration with respect to q_0 and the magnitude of $\mathbf{q}$ can now be performed, leaving one with an expression for w_1 in the form of an integral extended over all directions of the scattered electron, and all light quanta, as restricted by energy conservation. On averaging with respect to the polarization of the incident electron, and specializing to the Coulomb field of a nucleus, one obtains

$$w_1=\frac{\alpha}{\pi^2}|S^{(inc)}|\int_{k_0>0}d\Omega(dk)\delta(k^2)\frac{|\mathbf{q}|}{|\mathbf{p}|}\left[\frac{Z\alpha}{(\mathbf{p}-\mathbf{q}-\mathbf{k})^2}\right]^2\tfrac{1}{4}Tr\left[(i\gamma p-\kappa)\right.$$
$$\left.\times\left(\gamma_4\frac{i\gamma(q+k)-\kappa}{2qk}\gamma_\lambda-\gamma_\lambda\frac{i\gamma(p-k)-\kappa}{2pk}\gamma_4\right)(i\gamma q-\kappa)\left(\gamma_4\frac{i\gamma(p-k)-\kappa}{2pk}\gamma_\lambda-\gamma_\lambda\frac{i\gamma(q+k)-\kappa}{2qk}\gamma_4\right)\right]. \quad (2.77)$$

It may then be inferred that the differential cross section for radiative scattering through the angle ϑ, in which the energy loss does not exceed ΔE, is

$$\frac{d\sigma_1(\vartheta,\Delta E)}{d\Omega}=\frac{\alpha}{\pi^2}\int_{k_0=0}^{k_0=K}(dk)\delta(k^2)\frac{|\mathbf{q}|}{|\mathbf{p}|}\left[\frac{Z\alpha}{(\mathbf{p}-\mathbf{q}-\mathbf{k})^2}\right]^2\tfrac{1}{4}Tr\left[(i\gamma p-\kappa)\right.$$
$$\left.\times\left(\gamma_4\left(\frac{q_\lambda}{qk}-\frac{p_\lambda}{pk}\right)+\gamma_4\frac{\gamma k}{2qk}\gamma_\lambda+\gamma_\lambda\frac{\gamma k}{2pk}\gamma_4\right)(i\gamma q-\kappa)\left(\gamma_4\left(\frac{q_\lambda}{qk}-\frac{p_\lambda}{pk}\right)+\gamma_\lambda\frac{\gamma k}{2qk}\gamma_4+\gamma_4\frac{\gamma k}{2pk}\gamma_\lambda\right)\right], \quad (2.78)$$

where

$$K=\Delta E/\hbar c. \quad (2.79)$$

We shall first consider the simple situation in which the emitted radiation exerts a negligible reaction on the electron. That is to say, we shall treat the essentially elastic scattering of an electron, in which only a small fraction of the electron kinetic energy is radiated. Under these circumstances, which are expressed by $\Delta E\ll W=E-mc^2$, (2.79) simplifies to

$$\frac{d\sigma_1(\vartheta,\Delta E)}{d\Omega}=\left(\frac{Z\alpha}{2|\mathbf{p}|\beta}\csc^2\frac{\vartheta}{2}\right)^2\left(1-\beta^2\sin^2\frac{\vartheta}{2}\right)\frac{\alpha}{2\pi^2}\int_{k_0=0}^{k_0=K}(dk)\delta(k^2)\left(\frac{p}{pk}-\frac{q}{qk}\right)^2. \quad (2.80)$$

Now

$$\left(\frac{p}{pk}-\frac{q}{qk}\right)^2=\frac{(p-q)^2}{(pk)(qk)}+\kappa^2\left(\frac{2}{(pk)(qk)}-\frac{1}{(pk)^2}-\frac{1}{(qk)^2}\right), \quad (2.81)$$

and

$$\frac{1}{(pk)(qk)}=\frac{1}{(q-p)k}\left(\frac{1}{pk}-\frac{1}{qk}\right)=\frac{1}{2}\int_{-1}^{1}\frac{dv}{\left[\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)k\right]^2}, \quad (2.82)$$

from which one deduces, on integration by parts, that

$$\frac{2}{(pk)(qk)}-\frac{1}{(pk)^2}-\frac{1}{(qk)^2}=-\int_{-1}^{1}dv\,v\frac{\partial}{\partial v}\frac{1}{\left[\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)k\right]^2}. \quad (2.83)$$

Therefore,

$$\int (dk)\delta(k^2)\left(\frac{p}{pk}-\frac{q}{qk}\right)^2=\kappa^2\int_{-1}^{1}dv\left(\frac{(p-q)^2}{2\kappa^2}-v\frac{\partial}{\partial v}\right)\int(dk)\frac{\delta(k^2)}{\left[\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)k\right]^2}. \tag{2.84}$$

The k integration in the latter equation can be written as

$$-\int(dk)\frac{\delta(k^2)}{\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)^2}\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)_\lambda\frac{\partial}{\partial k_\lambda}\frac{1}{\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)k}$$

$$=\frac{1}{\kappa^2}\frac{1}{1+[(p-q)^2/4\kappa^2](1-v^2)}\left[\int(dk)\frac{\partial}{\partial k_\lambda}\left[\delta(k^2)\frac{\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)_\lambda}{\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)k}\right]-2\int(dk)\delta'(k^2)\right]. \tag{2.85}$$

However,

$$-2\int(dk)\delta'(k^2)=\int\frac{(dk)}{k_0}\frac{\partial}{\partial k_0}\delta(k^2)=\int(dk)\frac{\partial}{\partial k_0}\left(\frac{\delta(k^2)}{k_0}\right)+\int\frac{(dk)}{k_0^2}\delta(k^2), \tag{2.86}$$

so that

$$\int(dk)\delta(k^2)\left(\frac{p}{pk}-\frac{q}{qk}\right)^2=\int_{-1}^{1}dv\left(\frac{(p-q)^2}{2\kappa^2}-v\frac{\partial}{\partial v}\right)\frac{1}{1+[(p-q)^2/4\kappa^2](1-v^2)}\left[\int\frac{(dk)}{k_0^2}\delta(k^2)\right.$$

$$\left.+\int(dk)\frac{\partial}{\partial k_0}\left\{\delta(k^2)\left[\frac{\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)_0}{\left(\frac{p+q}{2}+\frac{p-q}{2}v\right)k}+\frac{1}{k_0}\right]\right\}\right], \tag{2.87}$$

in which we have discarded terms that obviously vanish on integration over the domain $0<k_0<K$.

The first bracketed integral in (2.87), when expressed in three-dimensional notation, becomes

$$\int\frac{(d\mathbf{k})dk_0}{k_0^2}\delta(\mathbf{k}^2-k_0^2)=2\pi\int_{k_{\min}}^{k}\frac{dk_0}{k_0}=2\pi\log\frac{K}{k_{\min}}, \tag{2.88}$$

in which we have again introduced an invariant minimum light quantum wave number to characterize a logarithmic divergence associated with the "infra-red catastrophe." A similar expression of the second bracketed integral in (2.87) yields.

$$\int(d\mathbf{k})\delta(\mathbf{k}^2-K^2)\left[\frac{1}{K}-\frac{p_0}{p_0K-\left(\frac{\mathbf{p}+\mathbf{q}}{2}+\frac{\mathbf{p}-\mathbf{q}}{2}v\right)\cdot\mathbf{k}}\right]$$

$$=2\pi\left[1-\frac{1}{2}\frac{p_0}{(\mathbf{p}^2-[(\mathbf{p}-\mathbf{q})^2/4](1-v^2))^{\frac{1}{2}}}\log\frac{p_0+(\mathbf{p}^2-[(\mathbf{p}-\mathbf{q})^2/4](1-v^2))^{\frac{1}{2}}}{p_0-(\mathbf{p}^2-[(\mathbf{p}-\mathbf{q})^2/4](1-v^2))^{\frac{1}{2}}}\right]. \tag{2.89}$$

Therefore,

$$\int_{k_0=0}^{k_0=K}(dk)\delta(k^2)(p/pk-q/qk)^2$$

$$=4\pi\int_0^1 dv\left(\frac{(\mathbf{p}-\mathbf{q})^2}{2\kappa^2}-v\frac{\partial}{\partial v}\right)\frac{1}{1+[(\mathbf{p}-\mathbf{q})^2/4\kappa^2](1-v^2)}\left[\log\frac{K}{k_{\min}}+1-\frac{1}{2\beta\xi}\log\frac{1+\beta\xi}{1-\beta\xi}\right], \tag{2.90}$$

where

$$\xi=\left(1-\sin^2\frac{\vartheta}{2}(1-v^2)\right)^{\frac{1}{2}}. \tag{2.91}$$

We may now employ the identity

$$\frac{1-1/2\beta\xi\log(1+\beta\xi/1-\beta\xi)}{1+[(\mathbf{p}-\mathbf{q})^2/4\kappa^2](1-v^2)}=\frac{1}{2}\frac{\log(1+[(\mathbf{p}-\mathbf{q})^2/4\kappa^2](1-v^2))}{1+[(\mathbf{p}-\mathbf{q})^2/4\kappa^2](1-v^2)}-\frac{\log(2p_0/\kappa)-1}{1+[(\mathbf{p}-\mathbf{q})^2/4\kappa^2](1-v^2)}$$
$$+\frac{\kappa^2}{p_0^2}\frac{1}{2\beta\xi}\left[\frac{\log\frac{1-\beta\xi}{2}}{1+\beta\xi}-\frac{\log\frac{1+\beta\xi}{2}}{1-\beta\xi}\right] \tag{2.92}$$

to cast (2.90) into the form

$$\int_{k_0=0}^{k_0=K}(dk)\delta(k^2)\left(\frac{p}{pk}-\frac{q}{qk}\right)^2=4\pi\frac{\mathbf{p}^2}{\kappa^2}\sin^2\frac{\vartheta}{2}\left[\left(\log\frac{K}{k_{\min}}-\log\frac{2p_0}{\kappa}+1\right)(F_0+F_1)+F_1+\tfrac{1}{2}G+H\right], \tag{2.93}$$

where

$$H=\left(1+\frac{1}{2\lambda^2}\right)\frac{\kappa^2}{p_0^2}\int_0^1\frac{dv}{\beta\xi}\left[\frac{\log\frac{1-\beta\xi}{2}}{1+\beta\xi}-\frac{\log\frac{1+\beta\xi}{2}}{1-\beta\xi}\right]-\frac{1}{2\lambda^2}\frac{\kappa^2}{p_0^2}\frac{1}{\beta}\left[\frac{\log\frac{1-\beta}{2}}{1+\beta}-\frac{\log\frac{1+\beta}{2}}{1-\beta}\right]. \tag{2.94}$$

The function H approaches a constant in the limit of small velocities,

$$\beta\ll 1:\quad H=\frac{4}{3}(\log 2-1)-\frac{1}{9}. \tag{2.95}$$

At high energies, H is approximated by

$$p_0/\kappa\gg 1:\quad H=-\frac{\kappa^2}{p_0^2}f(\vartheta), \tag{2.96}$$

with

$$f(\vartheta)=\frac{1}{\sin\vartheta/2}\int_{\cos\vartheta/2}^1\left(\frac{\log\frac{1+\xi}{2}}{1-\xi}-\frac{\log\frac{1-\xi}{2}}{1+\xi}\right)\frac{d\xi}{(\xi^2-\cos^2(\vartheta/2))^{\frac{1}{2}}}. \tag{2.97}$$

This integral can be performed analytically for $\vartheta=\pi$,

$$f(\pi)=\pi^2/12, \tag{2.98}$$

but must be evaluated numerically for other angles. An approximation in excess, which has the correct asymptotic form at small angles, is provided by

$$f(\vartheta)\sim\left(\frac{2}{\cos(\vartheta/2)(1+\cos(\vartheta/2))^2}\right)^{\frac{1}{2}}\left[\log\frac{1}{2(1-\cos(\vartheta/2))}+\frac{1-\cos(\vartheta/2)}{2}+1\right]. \tag{2.99}$$

This formula is reasonably accurate even for $\vartheta=\pi/2$, where the value yielded by (2.99) exceeds by only 8.6 percent the following result of a numerical calculation:

$$f(\pi/2)=1.167. \tag{2.100}$$

The total differential cross section for scattering through the angle ϑ, in which the energy loss does not exceed ΔE, is

$$\frac{d\sigma(\vartheta,\Delta E)}{d\Omega}=\frac{d\sigma_0(\vartheta)}{d\Omega}+\frac{d\sigma_1(\vartheta,\Delta E)}{d\Omega}=\left(\frac{Z\alpha}{2|\mathbf{p}|\beta}\csc^2\frac{\vartheta}{2}\right)^2\left(1-\beta^2\sin^2\frac{\vartheta}{2}\right)(1-\delta(\vartheta,\Delta E)), \tag{2.101}$$

where $\delta(\vartheta,\Delta E)$ is the desired fractional decrease in the cross section produced by radiative effects. For essentially elastic scattering, we obtain

$$\delta(\vartheta,\Delta E\ll W)=\frac{2\alpha}{\pi}\beta^2\frac{p_0^2}{\kappa^2}\sin^2\frac{\vartheta}{2}\left[\left(\log\frac{E}{\Delta E}-1\right)(F_0+F_1)+\frac{1}{2}F_0-F_1+\frac{1}{3}F_2-H+\frac{1}{2}\frac{\kappa^2/p_0^2}{1-\beta^2\sin^2(\vartheta/2)}F_0\right], \tag{2.102}$$

on combining (2.70) with (2.80). It will be noted that the infra-red catastrophe, as characterized by k_{min} has disappeared. However, it is possible, in principle, to consider the limit $\Delta E\to 0$, which would make δ diverge logarithmically. It is well known that this difficulty stems from the neglect of processes involving more than one low frequency quantum.[5] Actually, the essentially elastic scattering cross section must approach zero as $\Delta E\to 0$; that is, it never happens that a scattering event is unaccompanied by the emission of quanta. This is described by replacing the radiative correction factor $1-\delta$ with $e^{-\delta}$, which has the proper limiting behavior as $\Delta E\to 0$. The further terms in the series expansion of $e^{-\delta}$ express the effects of higher order processes involving the multiple emission of soft quanta. However, for practical purposes, such a refinement is unnecessary. The accuracy with which the energy of a particle can be measured is such that the limit $\Delta E\to 0$ cannot be realized, and δ will be small in comparison with unity under presently accessible circumstances.

For a slowly moving particle,

$$\beta\ll 1:\delta(\vartheta,\Delta E\ll W)=\frac{8\alpha}{3\pi}\beta^2\sin^2\frac{\vartheta}{2}\left[\log\frac{mc^2}{2\Delta E}+\frac{19}{30}\right], \qquad (2.103)$$

according to (2.95), the limiting form of H and the corresponding limiting form of F_n:

$$\lambda\ll 1:\quad F_n=\frac{1}{2n+1}. \qquad (2.104)$$

The radiative correction thus increases linearly with the kinetic energy of the particle. In the extreme relativistic region, on the other hand,

$$\frac{p_0}{\kappa}\sin\frac{\vartheta}{2}\gg 1:$$

$$\delta(\vartheta,\Delta E\ll W)=\frac{4\alpha}{\pi}\left[\left(\log\frac{E}{\Delta E}-\frac{13}{12}\right)\left(\log\frac{2p_0}{K}\sin\frac{\vartheta}{2}-\frac{1}{2}\right)+\frac{17}{72}+\frac{1}{2}\sin^2\frac{\vartheta}{2}f(\vartheta)\right], \qquad (2.105)$$

which has a logarithmic dependence on the particle energy. The asymptotic form (2.105) is quite accurate for even moderate energies. Thus, with $\vartheta=\pi/2$, $\Delta E=10$ kev and $W=3.1$ Mev, which corresponds to $(p_0/\kappa)\times\sin\vartheta/2=5$, the value of δ computed from (2.105) differs from the correct value,

$$\delta=8.6\ 10^{-2}, \qquad (2.106)$$

by only a fraction of a percent. It is evident from this numerical result that radiative corrections to scattering cross sections can be quite appreciable. For the particular conditions chosen, ΔE can be materially increased (but still subject to $\Delta E\ll W$), without seriously impairing δ. Thus, with $\Delta E=40$ kev, $\delta=6.3\ 10^{-2}$, while $\Delta E=80$ kev yields $\delta=5.1\ 10^{-2}$. As to the energy dependence of δ, we remark that with a given accuracy in the determination of the energy, $\Delta E/E$, δ varies linearly with the logarithm of the energy. Thus, with $\Delta E/E=0.04/3.6=1.1\ 10^{-2}$, an increase in the total energy by a factor of four produces an addition of $4.4\ 10^{-2}$ to δ, whence $\delta=11\ 10^{-2}$ for $W=14$ Mev, and $\delta=15\ 10^{-2}$ for $W=57$ Mev.

The angular dependence of δ at relativistic energies is not fully described by the asymptotic formula (2.105), since the underlying condition, $(p_0/\kappa)\sin\vartheta/2\ll 1$, cannot be maintained with diminishing ϑ. Indeed, δ is proportional to $\sin^2\vartheta/2$ at angles such that $(p_0/\kappa)\sin\vartheta/2\ll 1$. However (2.105) can be used over a wide angular range, even at moderate energies. Thus, with $W=3.1$ Mev, $\Delta E=40$ kev, and $\vartheta=\pi/4$, which corresponds to $(p_0/\kappa)\sin\vartheta/2=2.7$, the value of δ deduced from (2.105) exceeds by only 2 percent the correct value, $\delta=4.2\ 10^{-2}$. We may note that under the same energy conditions, but with $\vartheta=3\pi/4$, $\delta=7.2\ 10^{-2}$. The angular dependence of δ may be particularly suitable for an experimental test of these predictions, which involve the relativistic aspects of the radiative corrections to the electromagnetic properties of the electron.

We have thus far considered only the essentially elastic scattering of an electron, in which radiative corrections arise primarily from virtual processes. If we wish to compute the differential cross section for scattering with an arbitrary maximum energy loss ΔE, it is only necessary to augment the essentially elastic cross section, in which the maximum energy loss is $\Delta E'\ll W$, by the cross section for scattering with the emission of a light quantum in the energy range from $\Delta E'$ to ΔE. The latter process involves the well-known bremsstrahlung cross section which, of course, is the content of (2.79). This will be illustrated by the calculation of the differential cross section for the scattering of a slowly moving electron, irrespective of the final energy. The differential cross section per unit solid angle for scattering of an electron through the angle ϑ, in which a light quantum is emitted in the energy range from $\Delta E'$ to W, is

$$\frac{\alpha}{\pi^2}\int_{\Delta E'/\hbar c}^{W/\hbar c}\frac{dk_0}{k_0}\int d\omega\frac{|\mathbf{q}|}{|\mathbf{p}|}\left[\frac{Z\alpha}{(\mathbf{p}-\mathbf{q})^2}\right]^2[(\mathbf{p}-\mathbf{q})^2-(\mathbf{n}\cdot(\mathbf{p}-\mathbf{q}))^2], \qquad (2.107)$$

according to the non-relativistic limit of (2.78). Here $d\omega$ is an element of solid angle associated with the direction of the unit vector $\mathbf{n}=\mathbf{k}/k_0$, and

$$|\mathbf{q}|=(\mathbf{p}^2-2\kappa k_0)^{\frac{1}{2}}. \qquad (2.108)$$

[5] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

On performing the integration over all emission directions of the light quantum, and introducing the new variable of integration, $x=|\mathbf{q}|/|\mathbf{p}|$, (2.107) becomes

$$\frac{8\alpha}{3\pi}\left(\frac{Z\alpha}{|\mathbf{p}|}\right)^2\int_0^{(1-\Delta E'/W)^{\frac{1}{2}}}\frac{x}{1+x^2-2x\cos\vartheta}\frac{2xdx}{1-x^2} =\left(\frac{Z\alpha}{2|\mathbf{p}|\beta}\csc^2\frac{\vartheta}{2}\right)^2\frac{8\alpha}{3\pi}\beta^2\sin^2\frac{\vartheta}{2}\left[\log\frac{W}{\Delta E'}-\int_0^1\frac{1-x}{1+x^2-2x\cos\vartheta}\frac{2xdx}{1+x}\right]. \quad (2.109)$$

Thus the contribution to δ produced by emission of quanta with energies in the range from $\Delta E'$ to W is

$$-\frac{8\alpha}{3\pi}\beta^2\sin^2\frac{\vartheta}{2}\left[\log\frac{4W}{\Delta E'}-(\pi-\vartheta)\tan\frac{\vartheta}{2}-\frac{\cos\vartheta}{\cos^2\vartheta/2}\log\csc\frac{\vartheta}{2}\right]. \quad (2.110)$$

On adding this to $\delta(\vartheta,\Delta E')$, as given by (2.103), we obtain the desired result:

$$\beta\ll 1:\quad \delta(\vartheta,W)=\frac{8\alpha}{3\pi}\beta^2\sin^2\frac{\vartheta}{2}\left[\log\frac{1}{4\beta^2}+\frac{19}{30}+(\pi-\vartheta)\tan\frac{\vartheta}{2}+\frac{\cos\vartheta}{\cos^2\vartheta/2}\log\csc\frac{\vartheta}{2}\right]. \quad (2.111)$$

It may be remarked, finally, that the analogous meso-nuclear phenomenon, the radiative correction to nucleon-nucleon scattering associated with virtual meson emission, will be a relatively more significant effect in view of the stronger couplings involved. This may well be the explanation of the discrepancy between the observed neutron-proton scattering cross section for high energy neutrons and the larger theoretical values computed from various assumed interaction potentials.[6]

APPENDIX

In this section, we shall first give an alternative treatment of the polarization of the vacuum by an external field, employing the methods developed in the preceding pages. It is desired to compute the expectation value of $j_\mu(x)$,

$$\langle j_\mu(x)\rangle=(\Psi[\sigma],j_\mu(x)\Psi[\sigma]), \quad (A.1)$$

where $\Psi[\sigma]$ obeys

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=-\frac{1}{c}j_\mu(x)A_\mu(x)\Psi[\sigma], \quad (A.2)$$

and $A_\mu(x)$ is the potential of a prescribed current distribution. The physical situation can be described as follows. In the remote past, the matter field is uncoupled from the external electromagnetic field, and the state vector is that of the vacuum,

$$\Psi[-\infty]=\Psi_0. \quad (A.3)$$

It is supposed that the coupling is adiabatically switched on, and that the external field does not induce real pair creation. The latter restriction implies that the final state of the matter field, after the coupling is adiabatically switched off, is simply Ψ_0, whence

$$\Psi[\infty]-\Psi[-\infty]=(S-1)\Psi_0=0. \quad (A.4)$$

A solution of (A.2), in the form

$$\Psi[\sigma]=U[\sigma]\Psi_0, \quad (A.5)$$

may be constructed, where

$$i\hbar c\frac{\delta U[\sigma]}{\delta\sigma(x)}=-\frac{1}{c}j_\mu(x)A_\mu(x)U[\sigma], \quad (A.6)$$

and

$$U[\infty]=S,\quad U[-\infty]=1. \quad (A.7)$$

The current induced in the vacuum is then written as

$$\langle j_\mu(x)\rangle=\langle U^{-1}[\sigma]j_\mu(x)U[\sigma]\rangle_0. \quad (A.8)$$

Now

$$\tfrac{1}{2}\int_{-\infty}^{\infty}d\omega'\epsilon[\sigma,\sigma']\frac{\delta}{\delta\sigma'(x')}U^{-1}[\sigma']j_\mu(x)U[\sigma']=U^{-1}[\sigma]j_\mu(x)U[\sigma]-\tfrac{1}{2}(j_\mu(x)+S^{-1}j_\mu(x)S), \quad (A.9)$$

whence

$$U^{-1}[\sigma]j_\mu(x)U[\sigma]=\tfrac{1}{2}(j_\mu(x)+S^{-1}j_\mu(x)S)+\frac{i}{2\hbar c^2}\int_{-\infty}^{\infty}d\omega'\epsilon[\sigma,\sigma']U^{-1}[\sigma'][j_\mu(x),j_\nu(x')]U[\sigma']A_\nu(x'). \quad (A.10)$$

On placing $U[\sigma']=1$ on the right side of (A.10), one obtains the first approximation in a treatment that regards the disturbance of the vacuum as small. Hence,

$$\langle j_\mu(x)\rangle=\frac{i}{2\hbar c^2}\int_{-\infty}^{\infty}d\omega'\epsilon(x-x')[j_\mu(x),j_\nu(x')]_0A_\nu(x'), \quad (A.11)$$

in view of (A.4) and the absence of a current in the unperturbed vacuum. We shall, for convenience, write this formula as

$$\langle j_\mu(x)\rangle=\frac{4e^2}{\hbar}\int G_{\mu\nu}(x-x')A_\nu(x')d\omega', \quad (A.12)$$

where, according to II (2.10),

$$G_{\mu\nu}(x-x')=\tfrac{1}{4}Tr[S^{(1)}(x'-x)\gamma_\mu\bar{S}(x-x')\gamma_\nu+\bar{S}(x'-x)\gamma_\mu S^{(1)}(x-x')\gamma_\nu]. \quad (A.13)$$

The introduction of the Fourier integral representations for the functions $S^{(1)}$ and $\bar{S}$, combined with the trace evaluation (see II (2.10))

$$\tfrac{1}{4}Tr[(-i\gamma k'+\kappa)\gamma_\mu(i\gamma k''+\kappa)\gamma_\nu+(-i\gamma k''+\kappa)\gamma_\mu(i\gamma k'+\kappa)\gamma_\nu]=k_\mu'k_\nu''+k_\nu'k_\mu''-\delta_{\mu\nu}(k'k''-\kappa^2), \quad (A.14)$$

[6] A summary is given by L. Rosenfeld, Nuclear Forces (Interscience Publishers, Inc., New York, 1949), pp. 450, 454.

yields the following expression for $G_{\mu\nu}(x)$,

$$G_{\mu\nu}(x)=\frac{1}{(2\pi)^7}\int(dk')(dk'')e^{i(k'+k'')x}\frac{\delta(k''^2+\kappa^2)}{k'^2+\kappa^2}[k_\mu'k_\nu''+k_\nu'k_\mu''-\delta_{\mu\nu}(k'k''-\kappa^2)]. \tag{A.15}$$

It is instructive to examine the conditions imposed on $G_{\mu\nu}(x)$ by the related requirements of charge conservation and gauge invariance. The former evidently demands that

$$\frac{\partial}{\partial x_\mu}G_{\mu\nu}(x)=0. \tag{A.16}$$

The requirement of gauge invariance is that the induced current be unaffected by the gauge transformation

$$A_\mu(x)\to A_\mu(x)-\frac{\partial\Lambda(x)}{\partial x_\mu}, \tag{A.17}$$

or that

$$\int G_{\mu\nu}(x-x')\frac{\partial\Lambda(x')}{\partial x_\nu'}d\omega'=\int\frac{\partial}{\partial x_\nu}G_{\mu\nu}(x-x')\Lambda(x')d\omega'=0, \tag{A.18}$$

in which the absence of an integrated term is a consequence of the adiabatic removal of the coupling in the remote past and future. Evidently (A.18) is satisfied in virtue of (A.16), since

$$G_{\mu\nu}(x)=G_{\nu\mu}(x). \tag{A.19}$$

On computing $\partial G_{\mu\nu}(x)/\partial x_\mu$ from (A.15), we obtain

$$\frac{\partial}{\partial x_\mu}G_{\mu\nu}(x)=\frac{i}{(2\pi)^7}\int(dk')(dk'')e^{i(k'+k'')x}k_\nu''\delta(k''^2+\kappa^2)$$
$$=\frac{i}{(2\pi)^7}\int(dk)e^{ikx}\int(dk'')k_\nu''\delta(k''^2+\kappa^2), \tag{A.20}$$

where

$$k_\mu=k_\mu'+k_\mu'', \tag{A.21}$$

which is indeed zero if

$$\int(dk'')k_\nu''\delta(k''^2+\kappa^2)=0. \tag{A.22}$$

Although the latter integral is strictly divergent, the value of zero is unambiguously obtained from any limiting process in which the delta-function is replaced by a suitable non-singular function. In this sense, the requirements of charge conservation and gauge invariance are satisfied. It may be noted that the same integral is encountered in evaluating the current in the unperturbed vacuum, II (1.73),

$$\langle j_\mu(x)\rangle_0=\frac{iec}{2}Tr\gamma_\mu S^{(1)}(0)$$
$$=\frac{2ec}{(2\pi)^3}\int(dk'')k_\mu''\delta(k''^2+\kappa^2), \tag{A.23}$$

which must also be zero.

We return to the evaluation of $G_{\mu\nu}(x)$, and utilize the identity

$$k'k''-\kappa^2=2\frac{(kk')(kk'')}{k^2}-(k'^2+\kappa^2)\frac{kk''}{k^2}-(k''^2+\kappa^2)\frac{kk'}{k^2} \tag{A.24}$$

to simplify (A.15). The third term in the latter expression makes no contribution in view of the null value of $(k''^2+\kappa^2)\delta(k''^2+\kappa^2)$, while the second term produces a contribution of $G_{\mu\nu}(x)$ of the form.

$$\int(dk)(dk'')e^{ikx}\frac{kk''}{k^2}\delta(k''^2+\kappa^2), \tag{A.25}$$

which must also be zero, in consequence of (A.22). Hence,

$$G_{\mu\nu}(x)=\frac{1}{(2\pi)^7}\int(dk')(dk'')e^{ikx}\left(\frac{\delta(k'^2+\kappa^2)}{k''^2+\kappa^2}+\frac{\delta(k''^2+\kappa^2)}{k'^2+\kappa^2}\right)\left(\frac{k_\mu'k_\nu''+k_\nu'k_\mu''}{2}-\delta_{\mu\nu}\frac{(kk')(kk'')}{k^2}\right). \tag{A.26}$$

The delta-function factor can be simplified in the manner introduced in the text:

$$\frac{\delta(k'^2+\kappa^2)}{k''^2+\kappa^2}+\frac{\delta(k''^2+\kappa^2)}{k'^2+\kappa^2}=\frac{1}{k''^2-k'^2}(\delta(k'^2+\kappa^2)-\delta(k''^2+\kappa^2))=-\tfrac{1}{2}\int_{-1}^{1}dv\delta'\left(\frac{k'^2+k''^2}{2}+\frac{k'^2-k''^2}{2}v+\kappa^2\right). \tag{A.27}$$

The introduction of the new variables k_μ and p_μ, as defined by

$$k_\mu'=\tfrac{1}{2}k_\mu+\left(p_\mu-\frac{v}{2}k_\mu\right)$$
$$k_\mu''=\tfrac{1}{2}k_\mu-\left(p_\mu-\frac{v}{2}k_\mu\right), \tag{A.28}$$

then brings $G_{\mu\nu}(x)$ into the form

$$G_{\mu\nu}(x)=-\frac{1}{8(2\pi)^7}\int_{-1}^{1}dv(1-v^2)\int(dk)(dp)e^{ikx}(k_\mu k_\nu-\delta_{\mu\nu}k^2)\delta'\left(p^2+\kappa^2+\frac{k^2}{4}(1-v^2)\right), \tag{A.29}$$

where, in virtue of the dependence of the delta-function on p^2 alone, terms linear in p_μ have been discarded, and $p_\mu p_\nu$ has been replaced by $\tfrac{1}{4}\delta_{\mu\nu}p^2$. It is thereby shown that

$$G_{\mu\nu}(x)=\left(\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_\nu}-\delta_{\mu\nu}\Box^2\right)G(x), \tag{A.30}$$

with

$$G(x)=\frac{1}{8(2\pi)^7}\int_{-1}^{1}dv(1-v^2)\int(dk)(dp)e^{ikx}\delta'\left(p^2+\kappa^2+\frac{k^2}{4}(1-v^2)\right). \tag{A.31}$$

The divergent and convergent parts of $G(x)$ can be separated by a partial integration with respect to v,

$$G(x)=\frac{1}{6(2\pi)^3}\int(dp)\delta'(p^2+\kappa^2)\delta(x)-\Box^2\frac{1}{8(2\pi)^7}\int_0^1v^2\left(1-\frac{v^2}{3}\right)dv\int(dk)e^{ikx}\int(dp)\delta''\left(p^2+\kappa^2+\frac{k^2}{4}(1-v^2)\right). \tag{A.32}$$

The invariant, logarithmically divergent integral that occurs in the first term of (A.32) can be expressed in three-dimensional notation as

$$\int (dp)\delta'(p^2+\kappa^2) = -\int (d\mathbf{p})dp_0\frac{1}{2p_0}\frac{\partial}{\partial p_0}\delta(p_0^2-\mathbf{p}^2-\kappa^2) = -\tfrac{1}{4}\int\frac{(d\mathbf{p})}{(\mathbf{p}^2+\kappa^2)^{\frac{3}{2}}} = -2\pi\,\mathop{\mathrm{Lim}}_{P\to\infty}\left(\log\frac{P_0+P}{\kappa}-1\right), \tag{A.33}$$

where

$$P_0 = (P^2+\kappa^2)^{\frac{1}{2}}. \tag{A.34}$$

The convergent second integral of (A.32) is then obtained by differentiating (A.33) with respect to κ^2 and replacing the latter by $\kappa^2+(k^2/4)(1-v^2)$,

$$\int (dp)\delta''\left(p^2+\kappa^2+\frac{k^2}{4}(1-v^2)\right) = \frac{\pi}{\kappa^2+(k^2/4)(1-v^2)}. \tag{A.35}$$

With these evaluations, $G(x)$ becomes

$$G(x) = -\frac{1}{24\pi^2}\mathop{\mathrm{Lim}}_{P\to\infty}\left(\log\frac{P_0+P}{\kappa}-1\right)\delta(x)-\frac{1}{64\pi^2}\frac{1}{\kappa^2}\Box^2(F_1(x)-\tfrac{1}{3}F_2(x)), \tag{A.36}$$

where

$$F_n(x) = \int_0^1 v^{2n}dv\frac{1}{(2\pi)^4}\int (dk)\frac{e^{ikx}}{1+(k^2/4\kappa^2)(1-v^2)}. \tag{A.37}$$

Finally, we may insert (A.30) into (A.12) and integrate by parts to obtain

$$\langle j_\mu(x)\rangle = 16\pi\alpha\int G(x-x')J_\mu(x')d\omega', \tag{A.38}$$

where $J_\mu(x)$ is the current vector that generates the external electromagnetic field. The expression of $G(x)$ contained in (A.36) then yields

$$\langle j_\mu(x)\rangle = -\frac{2\alpha}{3\pi}\mathop{\mathrm{Lim}}_{P\to\infty}\left(\log\frac{P_0+P}{\kappa}-1\right)J_\mu(x)-\frac{\alpha}{4\pi}\frac{1}{\kappa^2}\Box^2\int (F_1(x-x')-\tfrac{1}{3}F_2(x-x'))J_\mu(x')d\omega', \tag{A.39}$$

in which the first term represents the logarithmically divergent renormalization of charge.

It should be remarked that the existence of a charge renormalization term would appear to contradict the conservation of charge, since it implies that a non-vanishing total charge is induced in the vacuum. Indeed, a formal evaluation of the total induced charge would yield zero,

$$\begin{aligned}\frac{1}{c}\int\langle j_\mu(x)\rangle d\sigma_\mu &= \frac{i}{2\hbar c^2}\int \epsilon(x-x')\left[\frac{1}{c}\int j_\mu(x)d\sigma_\mu, j_\nu(x')\right]_0 A_\nu(x')d\omega'\\ &= 0,\end{aligned} \tag{A.40}$$

since the operator of the total charge commutes with the current vector at an arbitrary point. The expression of $\langle j_\mu(x)\rangle$ as

$$\langle j_\mu(x)\rangle = \frac{\partial}{\partial x_\nu}\frac{4e^2}{\hbar}\int G(x-x')F_{\mu\nu}(x')d\omega', \tag{A.41}$$

where

$$F_{\mu\nu} = \frac{\partial}{\partial x_\mu}A_\nu-\frac{\partial}{\partial x_\nu}A_\mu, \tag{A.42}$$

is formally consistent with the result since

$$\begin{aligned}\frac{1}{c}\int\langle j_\mu(x)\rangle d\sigma_\mu &= \frac{2e^2}{\hbar}\int\left(d\sigma_\mu\frac{\partial}{\partial x_\nu}-d\sigma_\nu\frac{\partial}{\partial x_\mu}\right)\int G(x-x')F_{\mu\nu}(x')d\omega'\\ &= 0,\end{aligned} \tag{A.43}$$

in view of the theorem

$$\int\left(d\sigma_\mu\frac{\partial}{\partial x_\nu}F(x)-d\sigma_\nu\frac{\partial}{\partial x_\mu}F(x)\right) = 0. \tag{A.44}$$

However, it is evident that these formal manipulations are only justified if the integrand in (A.44) decreases sufficiently rapidly in space-like directions, which is not fulfilled for the field strengths generated by a charge distribution of non-vanishing total charge.

This difficulty can be avoided by treating the actual electromagnetic field as the limit of a spatially confined field, for which the total induced charge is zero. A convenient way to accomplish this is to introduce a finite light quantum mass, which is eventually allowed to vanish. We thus write the potentials generated by the given charge distribution as

$$A_\mu(x) = \frac{1}{c}\int \bar{D}(x-x')J_\mu(x')d\omega',\quad \frac{\partial A_\mu(x)}{\partial x_\mu} = 0, \tag{A.45}$$

where

$$\bar{D}(x) = \mathop{\mathrm{Lim}}_{\epsilon\to 0}\frac{1}{(2\pi)^4}\int\frac{e^{ikx}}{k^2+\epsilon^2}(dk). \tag{A.46}$$

The induced current can then be exhibited in the form

$$\begin{aligned}\langle j_\mu(x)\rangle &= -\frac{4e^2}{\hbar c}\int G(x-x')\Box'^2\bar{D}(x'-x'')J_\mu(x'')d\omega'd\omega''\\ &= \frac{4e^2}{\hbar c}\frac{1}{(2\pi)^4}\int e^{ik\xi}G(k)k^2\bar{D}(k)J_\mu(x-\xi)(dk)(d\xi),\end{aligned} \tag{A.47}$$

in which the second version involves the Fourier transforms of the functions, $G(x)$ and $\bar{D}(x)$,

$$\begin{aligned}G(k) &= \frac{1}{(4\pi)^2}\int_{-1}^1 dv(1-v^2)\int (dp)\delta'\left(p^2+\kappa^2+\frac{k^2}{4}(1-v^2)\right)\\ \bar{D}(k) &= \mathop{\mathrm{Lim}}_{\epsilon\to 0}\frac{1}{k^2+\epsilon^2}.\end{aligned} \tag{A.48}$$

The total induced charge can then be calculated in terms of the total external charge,

$$Q=\frac{1}{c}\int J_\mu(x-\xi)d\sigma_\mu, \tag{A.49}$$

which expression is independent of ξ. Therefore,

$$\begin{aligned}\frac{1}{c}\int\langle j_\mu(x)\rangle d\sigma_\mu&=\frac{4e^2}{\hbar c}Q\int G(k)k^2\bar{D}(k)\delta(k)(dk)\\&=\frac{4e^2}{\hbar c}Q\operatorname*{Lim}_{\epsilon\to 0}\left(\frac{k^2}{k^2+\epsilon^2}G(k)\right)_{k_\mu=0}.\end{aligned} \tag{A.50}$$

If ϵ is placed equal to zero before evaluating the Fourier transforms at $k_\mu=0$, we obtain the previously computed non-vanishing induced charge

$$\delta Q=\frac{4e^2}{\hbar c}Q[G(k)]_{k_\mu=0}=\frac{\alpha}{3\pi^2}Q\int(dp)\delta'(p^2+\kappa^2). \tag{A.51}$$

On the other hand, if the limiting process $\epsilon\to 0$ is reserved to the end of the calculation, we evidently find $\delta Q=0$.

The implications of this limiting process may be further indicated by noting that, in the first term of (A.39), $J_\mu(x)$ will be replaced by

$$J_\mu(x)-\epsilon^2 cA_\mu(x), \tag{A.52}$$

in virtue of the differential equation

$$(\square^2-\epsilon^2)A_\mu(x)=-\frac{1}{c}J_\mu(x) \tag{A.53}$$

obeyed by the potential (A.45). Now (A.52) reduces to $J_\mu(x)$ at any point as $\epsilon\to 0$. Yet the total charge computed from (A.52) is zero. This is illustrated by the charge density associated, according to (A.52), with a point charge at the origin:

$$\operatorname*{Lim}_{\epsilon\to 0}e\left(\delta(\mathbf{r})-\epsilon^2\frac{e^{-\epsilon r}}{4\pi r}\right). \tag{A.54}$$

We may conclude that in the process of vacuum polarization, a non-vanishing, and indeed divergent charge is attached to the original charge distribution, and a compensating charge is created at infinity.

We shall finally apply the computational methods of this paper to evaluate the invariant expression for the electromagnetic mass,

$$\delta mc^2\psi(x)=-\frac{e^2}{2}\int\gamma_\mu[\bar{D}(x-x')S^{(1)}(x-x')+D^{(1)}(x-x')\bar{S}(x-x')]\gamma_\mu\psi(x')d\omega'. \tag{A.55}$$

The insertion of the Fourier integral representations yields

$$\delta mc^2\psi(x)=-\frac{e^2}{2}\frac{1}{(2\pi)^7}\int(dk)(dk')e^{i(k+k')(x-x')}\gamma_\mu(i\gamma k'-\kappa)\gamma_\mu\left(\frac{\delta(k'^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{k'^2+\kappa^2}\right)\psi(x')d\omega'. \tag{A.56}$$

This becomes

$$-\frac{e^2}{2}\frac{1}{(2\pi)^7}\int(dk)(dp)e^{ip(x-x')}\gamma_\mu(i\gamma(p-k)-\kappa)\gamma_\mu\left(\frac{\delta(k^2-2pk)}{2pk}-\frac{\delta(k^2)}{2pk}\right)\psi(x')d\omega' \tag{A.57}$$

on introducing

$$p_\mu=k_\mu+k_\mu', \tag{A.58}$$

which is effectively subject to the restriction

$$p^2+\kappa^2=0, \tag{A.59}$$

in view of the wave equation satisfied by $\psi(x')$. Now

$$\frac{1}{2pk}[\delta(k^2-2pk)-\delta(k^2)]=-\int_0^1\delta'(k^2-2pku)du, \tag{A.60}$$

and

$$\gamma_\mu(i\gamma(p-k)-\kappa)\gamma_\mu=-2(i\gamma(p-k)+2\kappa), \tag{A.61}$$

whence

$$\delta mc^2\psi(x)=\frac{e^2}{(2\pi)^7}\int(dk)(dp)\int_0^1 du\,e^{ip(x-x')}(i\gamma k-\kappa)\delta'(k^2-2pku)\psi(x')d\omega', \tag{A.62}$$

in which we have employed the fact that $i\gamma p+\kappa$ is equivalent to $\gamma_\mu(\partial/\partial x_\mu')+\kappa$ applied to $\psi(x')$ and is thus effectively equal to zero. The transformation

$$k_\mu\to k_\mu+p_\mu u \tag{A.63}$$

then yields

$$\begin{aligned}\delta mc^2\psi(x)&=\frac{e^2}{(2\pi)^7}\int(dk)(dp)\int_0^1 du\,e^{ip(x-x')}(i\gamma pu-\kappa)\delta'(k^2+\kappa^2u^2)\psi(x')d\omega'\\&=-\frac{e^2}{(2\pi)^3}\kappa\int_0^1 du\int(dk)(1+u)\delta'(k^2+\kappa^2u^2)\psi(x).\end{aligned} \tag{A.64}$$

Hence,

$$\delta m/m=-\frac{\alpha}{2\pi^2}\int_0^1(1+u)du\int(dk)\delta'(k^2+\kappa^2u^2)=-\frac{3\alpha}{4\pi^2}\int(dk)\delta'(k^2+\kappa^2)+\frac{\alpha}{2\pi^2}\int_0^1(2u^2+u^3)du\kappa^2\int(dk)\delta''(k^2+\kappa^2u^2), \tag{A.65}$$

and

$$\delta m/m=\frac{3\alpha}{2\pi}\operatorname*{Lim}_{K\to\infty}\left(\log\frac{K_0+K}{\kappa}-\frac{1}{4}\right), \tag{A.66}$$

according to the integrals (A.33) and (A.35).